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اختيار النموذج وتقدير المعلمات للمعادلات التفاضلية التصادفية لمعدل انتشار فيروس نقص المناعة البشرية في العالم العربي

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المستخلص :

تنشأ الأنظمة الديناميكية في مجالات متنوعة، منها الاقتصاد والفيزياء والأحياء والهندسة، بالإضافة إلى الظواهر العشوائية. ونتيجةً لذلك، تُعد المعادلات التفاضلية التصادفية أداةً رياضيةً مهمةً لنمذجة الأنظمة الديناميكية في مختلف المجالات. في هذه الدراسة، درسنا طريقة الاختزال الخطي (التحويلي)، التي تُحوّل بعض المعادلات التفاضلية التصادفية غير الخطية إلى معادلات خطية باستخدام صيغة إيتو المتكاملة، ثم حددنا الحل التحليلي. استخدمنا طريقة الامكان الاعظم لتقدير المعاملات، مما يعكس البيانات الواقعية لمعدل انتشار فيروس نقص المناعة البشرية في العالم العربي من عام ١٩٩٠ إلى عام ٢٠٢٢. درسنا السلسلة الزمنية لانتشار فيروس نقص المناعة البشرية في العالم العربي من عام ١٩٩٠ إلى عام ٢٠٢٢، باستخدام نماذج غير خطية مثل نماذج المعادلات التفاضلية التصادفية . تم استخدام برنامج مابل لمقارنة النتائج.

Model Selection and Parameter Estimation for Stochastic Differential Equations for the Prevalence of HIV Average in the Arab World

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Abstract:

Dynamic systems arise in a variety of domains, including economics, physics, biology, and engineering, In addition to random phenomena. As a result, stochastic differential equations are an important mathematical tool for modeling dynamic systems in various fields. In this study, we examined the linear (transformation) reduction method, which transforms some non-linear stochastic differential equations(SDEs) into linear ones using Ito's integrated formula, and subsequently identified the analytical solution. We utilized the maximum likelihood estimation method to estimate the parameters, reflecting the realistic data of the HIV prevalence rate in the Arab world from 1990 - 2022. We study the time series of Human Immunodeficiency Virus (HIV) prevalence in the Arab world from 1990 - 2022, utilizing non-linear models such as SDE models . Maple software was used to compare the outcomes.

Keywords: stochastic differential equation(SDE) , Reducible, Ito's integral formula, maximum likelihood estimation method.

1- Introduction :

The Basic principles for modeling in science, engineering, and mathematics are deterministic models and stochastic differential equations (SDEs). It is possible to utilize more accurate differential equation models to solve more challenging issues as computing power rises. Two things might cause a model to be stochastic: either the data calibration suggests it is, as in financial mathematics, or when basic microscopic laws are coarse-grained, they produce stochastic behavior, similar to molecular dynamics in biology, chemistry, and materials science(Oksendal, 2013, p.62). Statistical mechanics commonly accounts for fluctuations by introducing a stochastic component into the deterministic differential equation. In statistical mechanics, it is common to introduce a stochastic term into a deterministic differential equation in order to account for fluctuations(Oksendal, 2013, p.62). A SDE is a type of differential equation that includes one or more stochastic processes as terms and has a solution that is also stochastic. Any undesired signal that contributes to the data is referred to as noise . In dynamical systems, noise is typically regarded as an annoyance . In the SDE, noise is the most significant factor (Gelbrich and Römisch, 1995,p.103-104). If a differential equation model is created, it is ideally possible to obtain an exact solution for the given physical phenomenon . However, this is typically not possible, even for ordinary differential equations (Mortensen, 1969,p.271-296).

2- Stochastic calculus:

Numerous physical applications require the treatment of random quantities that depend on a parameter. This phenomenon is known as Brownian motion. A parameter influences a random variable that determines the position of the Brownian particle for each coordinate (Mortensen, 1969 p.271-296). A stochastic process $X_t(w)$, provided as a family of random variables $\{X_t(w): t \in T, w \in \Omega\}$, is defined on the probability space (Ω, ς, P), where P is the probability measure defined on all elements of Ω, and ς is a σ-algebra of its subsets (Mortensen, 1969 p.271-296).

Stochastic calculus is a discipline of mathematics focused on studying stochastic processes It allows for the development of a consistent theory of integration for the integrals of stochastic processes in comparison to other stochastic processes(Rezaeyan and Baloui , 2013,p.63). The Wiener process is the most well-known stochastic process to which stochastic calculus is applied. Ito calculus and Stratonovich are essential components of stochastic calculus. The Ito calculus applies calculus techniques to stochastic phenomena, including Brownian motion. Below is the definition of an integral:

$$\int_0^T f(t)dt = \lim_{n\to+\infty} \sum_{j=1}^{n} f(\tau_j)\,(t_{j+1} - t_j) \tag{1}$$

where the value of τ_j is between $[t_j, t_{j+1}]$. Have the Riemann-Stieltjes integral more broadly:

$$\int_0^T f(t)dt = \lim_{n\to+\infty} \sum_{j=1}^{n} f(\tau_j)\,(g(t_{j+1}) - g(t_j)) \tag{2}$$

The limit of the smooth measure $g(t)$ converges to a unique value, regardless of the specific value of τ_j within the interval $[t_j, t_{j+1}]$.

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The Ito and Stratonovich calculus follows the same rules as for the regular Riemann-Stieltjes calculus. (Cyganowski, 2001,p.241). If we take the lower end point of the partition $[t_j, t_{j+1}]$, We have the Ito integral, but the Stratonovich case results if we select the midpoint $\frac{t_j+t_{j+1}}{2}$.

Let $t_0 = 0 \leq t_1 \leq t_2 \leq \cdots \leq t_n = T$ be a partition of the interval $[0, T]$ and$\delta = \max(t_i - t_{i-1})$. The Ito integral $\int_0^T \hbar(t, X_t) dW_t$ represents the limit in the quadratic mean .

$$\int_0^T \hbar(t, X_t) dW_t = \lim_{\delta_n \to 0} \sum_{i=1}^n \hbar(\tau_{i-1}, X_{\tau_{i-1}})(W_{t_i} - W_{t_{i-1}})$$

(3)

If the integrand z is jointly measurable and

$$\int_0^T E(|\hbar(s, X_s)|^2) ds < \infty$$

(4)

The probability limit in equation (2) is known as the stochastic integral. The definition of the Stratonovich integral is

$$\int_0^T \hbar(t, X_t) \circ dW_t = \lim_{\delta_n \to 0} \sum_{i=1}^n \hbar(\tau_{i-1}, \frac{X_{\tau_{i-1}} + X_{\tau_i}}{2})(W_{t_i} - W_{t_{i-1}})$$

(5)

The existence of the Stratonovich integral in equations (3) is contingent upon the continuity of the function $\hbar(t, X_t)$ in t and the continuity of its partial derivatives (see (Oksendal, 2013,p.64)- (Rezaeyan , Farnoush and Jamkhaneh ,2011,p.35-41)) . Moreover

In addition to the Ito integral's existence constraints, the existence of the Stratonovich integral in (3) requires the $\hbar(t, X_t)$ function to be continuous in t and have continuous partial derivatives (see (Oksendal, 2013,p.21-30)- (Rezaeyan , Farnoush and Jamkhaneh ,2011,p.35-41)) . Moreover

$$\int_0^T \hbar(t, X_t) \circ dW_t = \int_0^T \hbar(t, X_t) dW_t + \frac{1}{2} \int_0^T g(t, X_t) \frac{\partial h}{\partial x}(t, X_t) dt$$

(6)

or, equivalently(Mortensen, 1969,p.271-296)

$$\hbar(t, X_t) \circ dW_t = \hbar(t, X_t) dW_t + \frac{1}{2} g(t, X_t) \frac{\partial h}{\partial x}(t, X_t) dt$$

(7)

Consider a stochastic differential equation

$$dX_t = f(t, X_t)dt + g(t, X_t)dW_t$$
(8)

In this case, W_t is a p-dimensional Brownian motion process, often known as a Wiener process, and f , g and t are vector valued functions, $n \times p$ matrix valued functions . The process X_t represents the solution to the SDE (6) for all t in the interval [0, T].

Furthermore, we assume that X_t is distribution is well-known and unaffected by W_t . The Stratonovich interpretation of (6) is expressed via an explicit several-dimensional formula .

$$dX_t = \tilde{f}(t, X_t)dt + g(t, X_t) \circ dW_t$$
(9)

Where

$$\tilde{f}(t, X_t) = f(t, X_t) + \frac{1}{2}\sum_{j=1}^{p}\sum_{k=1}^{n}\frac{\partial g_{ij}}{\partial x_j}$$
(10)

Stochastic differential equations use the Ito integral and the Stratonovich . The . In this paper we study SDEs solved using Ito's formula integral following is a SDE :

$$X_t' = f(t, X_t) + g(t, X_t)\xi_t \, , X_0 = x_0 \, , t \geq 0$$
(11)

where ξ_t stands for a generalized stochastic process, f stands for the deterministic part, and $g\xi_t$ for the stochastic part (Oksendal, 2013,p.26-37)-(Mortensen, 1969,p.271-296) .

White noise is an illustration of an extended stochastic process. It is possible to define derivatives of any order for a generalized stochastic process (Evans , 2012,p35-36) . Let us assume that W_t is a generalized Wiener process that is employed to simulate the fluctuations in the prevalence of HIV in the Arab world.

With feature $W_t \sim N(0, t)$, $(0 \leq t \leq T)$, a Wiener process is a time-continuous process that is typically practically never differentiable. The

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٢٠٢٥م /١٤٤٧هـ

definition of white noise ξ_t is $\xi_t = \frac{dW_t}{dt}$ (Oksendal, 2013,p.36) . An Ito stochastic differential equation can be recast as follows if we swap out dW_t for $\xi_t dt$ in equation (11) :

$$dX_t = f(t, X_t)dt + g(t, X_t)dW_t \quad (12)$$

where X_t is the solution that we attempt to find using the integrating factor , and f and g stand for the drift and diffusion terms, respectively (Rezaeyan and Baloui , 2013,p.65).

2-1 Reducible Stochastic Differential Equations (Gelbrich and Römisch, 1995, p.113):

By employing the suitable substitution $X_t = U(t, Y_t)$, specific nonlinear SDEs:

$$dY_t = a(t, Y_t)dt + b(t, Y_t)dW_t \quad (13)$$

can be reduced to a linear SDE in X_t

$$dX_t = (a_1(t)X_t + a_2(t))dt + (b_1(t)X_t + b_2(t))dW_t \quad (14)$$

If $\frac{\partial U}{\partial y}(t, y) \neq 0$, the inverse function theorem ensures the existence of a local inverse $y = V(t, x)$ of $x = U(t, y)$, that is with $x = U\big(t, V(t, x)\big)$ and $y = V\big(t, U(t, y)\big)$. A solution Y_t

of (13) then has the form $Y_t = V(t, X_t)$ where X_t is given by (14) for appropriate coefficients a_1 , a_2 , b_1 and b_2 . from the Ito formula

$$dU(t, Y_t) = \left(\frac{\partial U}{\partial t} + a\frac{\partial U}{\partial y} + \frac{1}{2}b^2\frac{\partial^2 U}{\partial y^2}\right)dt + b\frac{\partial U}{\partial y}dW_t$$

where (t, Y_t) is the evaluation point for the coefficients and partial derivatives . A linear SDE of the form (14) corresponds with this if

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العدد الثاني والثلاثون
٢٠٢٥م /١٤٤٧هـ

$\frac{\partial U}{\partial t}(t,y)+a(t,y)\frac{\partial U}{\partial y}(t,y)+\frac{1}{2}b^2(t,y)\frac{\partial^2 U}{\partial y^2}(t,y)=a_1(t)U(t,y)+a_2(t)$
(15)

and
$b(t,y)\frac{\partial U}{\partial t}(t,y)=b_1(t)U(t,y)+b_2(t)$
(16)

There is not much else that can be stated that is clear-cut or precise at this level of generality . Specializing to the case where $a_1(t)\equiv b_1(t)\equiv 0$ and writing $a_2(t)=\alpha(t)$ and $b_2(t)=\beta(t)$ we obtain from (15) the identity

$$\frac{\partial^2 U}{\partial t\partial y}(t,y)=-\frac{\partial}{\partial y}\left(a(t,y)\frac{\partial U}{\partial y}(t,y)+\frac{1}{2}b^2(t,y)\frac{\partial^2 U}{\partial y^2}(t,y)\right)$$

and, from (16), the identities

$$\frac{\partial}{\partial y}\left(b(t,y)\frac{\partial U}{\partial t}(t,y)\right)=0$$

and

$$b(t,y)\frac{\partial^2 U}{\partial t\partial y}(t,y)+\frac{\partial b}{\partial t}(t,y)\frac{\partial U}{\partial t}(t,y)=\beta'(t)$$

Assume for now that $b(t,y)\neq 0$. Then, eliminating U and its derivatives we obtain

$$\beta'(t)=\beta(t)b(t,y)\left(\frac{1}{b^2(t,y)}\frac{\partial b}{\partial t}(t,y)-\frac{\partial}{\partial y}\left(\frac{a(t,y)}{b(t,y)}\right)++\frac{1}{2}\frac{\partial^2 b}{\partial y^2}(t,y)\right)$$

Given the left hand side's independence from y, this implies

$$\frac{\partial \gamma}{\partial y}(t,y)=0$$

where

$$\gamma(t,y)=\frac{1}{b(t,y)}\frac{\partial b}{\partial t}(t,y)-b(t,y)\frac{\partial}{\partial y}\left(\frac{a(t,y)}{b(t,y)}-\frac{1}{2}\frac{\partial b}{\partial y}(t,y)\right)$$

(17)

This is a sufficient condition for the reducibility of the nonlinear SDE (13) to the explicitly integrable SDE

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العدد الثاني والثلاثون
٢٠٢٥م /١٤٤٧هـ

$dX_t = \alpha(t)dt + \beta(t)dW_t$
(18)
through the modification $x = U(t,y)$. It can be found from (15) and (16), which reduce to in this particular instance .

$$\frac{\partial U}{\partial t}(t,y) + a(t,y)\frac{\partial U}{\partial y}(t,y) + \frac{1}{2}b^2(t,y)\frac{\partial^2 U}{\partial y^2}(t,y) = \alpha(t)$$

and

$$b(t,y)\frac{\partial U}{\partial t}(t,y) = \beta(t)$$

resulting in

$$U(t,y) = C\, e^{\left(\int_0^t \gamma(s,y)ds\right)} \int_0^y \frac{1}{b(t,y)} dz$$

Where C is an arbitrary constant in this case. We note that other linear SDEs can also be reduced to stochastic differentials of the form (18) using this method .

It is possible to reduce a nonlinear autonomous SDE using a variation of this process.
$dY_t = a(Y_t)dt + b(Y_t)dW_t$
(19)
to the autonomous linear SDE

$$dX_t = (a_1 X_t + a_2)dt + (b_1 X_t + b_2)dW_t$$
(20)

through $X_t = U(Y_t)$, a time-independent transformation . Here, the identities (15) and (16) have the following form :

$$a(y)\frac{\partial U}{\partial y}(y) + \frac{1}{2}b^2(y)\frac{\partial^2 U}{\partial y^2}(y) = a_1 U(y) + a_2$$
(21)

and

$$b(y)\frac{\partial U}{\partial t}(y) = b_1 U(y) + b_2$$
(22)

Assuming that $b(y) \neq 0$ and $b_1 \neq 0$, it follows from (22) that

$$U(y) = Ce^{(b_1 B(y))} - \frac{b_2}{b_1}$$
(23)

where

$$B(y)=\int_{y_0}^{y}\frac{ds}{b(s)}$$

and the constant C is arbitrary. This expression can be substituted for $U(y)$ in (21) to obtain

$$\left(b_1A(y)+\frac{1}{2}b_1^2-a_1\right)Ce^{(b_1B(y))}=a_2-a_1\frac{b_2}{b_1}$$

(24)

where

$$A(y)=\frac{a(y)}{b(y)}-\frac{1}{2}\frac{db}{dy}(y)$$

Differentiating (24), multiplying the result by $(b(y)e^{(b_1B(y))})/b_1$ and then differentiating again, we get

$$b_1\frac{dA}{dy}+\frac{d}{dy}\left(b\frac{dA}{dy}\right)=0$$

(25)

This is certainly satisfied if $\frac{dA}{dy}=0$ or if

$$\frac{d}{dy}\left(\frac{\frac{d}{dy}\left(b\frac{dA}{dy}\right)}{\frac{dA}{dy}}\right)=0$$

(26)

provided b_1 is chosen so that

$$b_1=-\frac{\frac{d}{dy}\left(b\frac{dA}{dy}\right)}{\frac{dA}{dy}}$$

If $b_1\neq 0$ the appropriate transformation is

$$U(y)=Ce^{(b_1B(y))}$$

(27)

whereas if $b_1=0$ it takes the form

$$U(y)=b_2B(y)+C$$

(28)

where b_2 is chosen so that (22) is satisfied .

3-Model Identification:

The time series of prevalence of HIV in Arab world for the years 1990-2022 will be studied, as shown in the figure (1) .

Figure(1) : prevalence of HIV, total (% of population ages 15–49)

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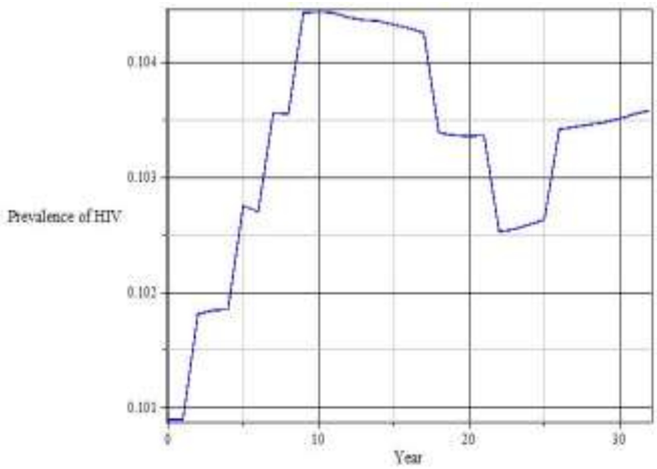

The time series for the prevalence of HIV, total (% of population ages 15–49), in the Arab World is seen in the above image to be increasing and decreasing erratically. This suggests that non-linear SDEs may be used .

4- Parameter Estimation for Stochastic Differential Equations :

It is aimed to present the parameter estimation techniques which are vital for SDE in this part. Stochastic Differential Equation having form of

$$dX_t = f(t, X_t; \theta)dt + g(t, X_t; \theta)dW_t \qquad (29)$$

is used to model the problem in our study. That is, we try to model the change in our variable with respect to time by using this equation. In this equation $\theta \in R$ is unknown parameter vectors . There is an assumption that $x_0, x_1, x_2, \ldots, x_N$ are observance of X_t and they are observed in the equally distributed times $t_i = i\Delta t$ for $i = 0,1,\ldots,N$ where $\Delta t = T/N$. In this study, the core idea is to estimate the θ vector by using data to model the problem in which we are interested . In this section, the maximum likelihood estimation method is presented (Allen , 2007,p.118) .

4-1 A maximum likelihood estimation method:

Given the vector θ, let $p(t_k, x_k | t_{k-1}, x_{k-1}; \theta)$ denote the transition probability density of (t_k, x_k) beginning from (t_{k-1}, x_{k-1}) . Assume that $p_0(x_k|\theta)$ represents the density of the initial state . In maximum likelihood estimation of θ [(Dennis, 1991,p.115-143), (Colombo, 2013) , (Hurn, 2003,p.45-63)], the joint density

$$D(\theta) = p_0(x_k|\theta)\prod_{k=1}^{N} p(t_k, x_k|t_{k-1}, x_{k-1}; \theta)$$

is increased throughout $\theta \in R_m$. This section will use the notation θ^* to represent the value of θ that maximizes $D(\theta)$. Nevertheless, it is more practical to minimize the function $L(\theta) = -\ln(D(\theta))$ using the following form in order to avoid small numbers on a computer :

$$L(\theta) = -\ln(p_0(x_k|\theta)) - \sum_{k=1}^{N} \ln(p(t_k, x_k|t_{k-1}, x_{k-1}; \theta))$$

The fact that the transition densities are not well understood is one challenge in determining the ideal value of θ^* . However, by considering the Euler approximation to (29) and letting $X_{k-1} = x_{k-1}$ at $t = t_{k-1}$, then

$$X_{t_k} = x_{k-1} + f(t_{k-1}, x_{k-1}; \theta)\Delta t + g(t_{k-1}, x_{k-1}; \theta)\sqrt{\Delta t}\eta_k$$

Where $\eta_k \sim N(0,1)$. This implies that

$$p(t_k, x_k|t_{k-1}, x_{k-1}; \theta) \approx \frac{1}{\sqrt{2\pi\sigma_k^2}} \exp\left(\frac{-(x_k - \mu_k)^2}{2\sigma_k^2}\right)$$

Where $\mu_k = x_{k-1} + f(t_{k-1}, x_{k-1}; \theta)\Delta t$ and $\sigma_k = g(t_{k-1}, x_{k-1}; \theta)\sqrt{\Delta t}$. You can replace this transition density in the expression for $L(\theta)$, and then minimize it over $\mathbb{R}^m$. Now, the second challenge is to minimize $L(\theta)$ in order to get the best vector θ^*. This is a nontrivial computation . For calculating the minimum of $L(\theta)$, the Nelder-Mead technique (Leader, 2004,), a numerical optimisation procedure, may be useful. . The Euler formula was used in the aforementioned technique to approximate the transition densities . As explained in (Hurn, 2003,p.45-63), one can approximate the transition density $p(t_k, x_k|t_{k-1}, x_{k-1}; \theta)$ through simulation as an alternative to utilizing Euler's formula . First, one can solve (29) numerically for a given value of θ by beginning at x_{k-1} and ending at t_{k-1} . One or more steps of a typical approach, like Milstein's or Euler's, can be applied . To get M estimated values $y_1, y_2, \dots y_M$ for X_t at $t = t_k$, this calculation is performed M times. Observe that the Wiener increments, which were kept in order to find $y_1, y_2, \dots y_M$. For every tested value of θ, the same Wiener increments are used to find $y_1, y_2, \dots y_M$ at t_k . This makes the task of determining the ideal value of θ^* deterministic . Next,

$$p^{(M)}(t_k, x_k | t_{k-1}, x_{k-1}; \theta) = \frac{1}{Mh} \sum_{j=1}^{M} k\left(\frac{x_k - y_j}{h}\right)$$

is used to estimate the transition density $p(t_k, x_k | t_{k-1}, x_{k-1}; \theta)$. where h is a bandwidth and K is a nonnegative kernel function. A sensible bandwidth h and kernel K are (Hurn, 2003,p.45-63):

$$K(z) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z^2}{2}\right) \text{ and } h = 0.9sM^{-1/5}$$

Where

$$s^2 = \frac{1}{M-1}\left[\sum_{j=1}^{M} y_j^2 - \frac{1}{M}\left(\sum_{j=1}^{M} y_i\right)^2\right]$$

provides the sample standard deviation formula. Next, one reduces

$$L^{(M)}(\theta) = -\ln\ (p_0(x_0|\theta)) - \sum_{k=1}^{N} \ln\ (p^{(M)}(t_k, x_k | t_{k-1}, x_{k-1}; \theta))$$

to find an approximation to the optimal value θ^* .

5- Modeling:

Figure (1) shows that the time series behaves increasing and decreasing in a non-linear manner, which leads us to use stochastic differential models or equations with limits of non-linear functions. Several mathematical models will be applied after finding them and estimating their parameters. An analytical solution will be formulated using the reduction method.

The first model: The nonlinear SDE

$$dX_t = \theta_1 X_t(1 - X_t)dt + \theta_2 X_t dW_t \quad (3\text{٠})$$

Where X_t is the prevalence of HIV, the initial value, $X_0 = 0.10089$, represents the first reading of the diffusion rate for this model. The average prevalence of HIV in the Arab world is given in Table 1 for 33 consecutive years. The model parameters (θ_1 and θ_2) will be estimated by the maximum likelihood estimation method . The following estimates will be obtained while estimating the parameters of the SDE of Equation (30) using the maximum likelihood method:

$$\theta_1 = \left(\frac{1}{\sum_{j=1}^{33}(1-X_j)^2\Delta t}\right)\left(\sum_{j=1}^{33}\frac{(1-X_j)(X_j-X_{j-1})}{X_j}\right) = 0.0188$$

$$\theta_2 = \sqrt{\frac{2\sum_{j=1}^{33}(X_j-X_{j-1})^2}{\sum_{j=1}^{33}(X_j^2-X_{j-1}^2)}} = 0.1621$$

So the model obtained in this application is

$dX_t = 0.0188X_t(1-X_t)dt +$
$0.1621X_tdW_t$ (3١)

To find the general solution of Equation (3١) by the reducible method . where $a(t,X_t) = 0.0188X_t(1-X_t)$ and $b(t,y) = 0.1621X_t$, so $A \equiv 0.0348 - 0.116X_t$. Thus (25) is satisfied with any b_1 . For $b_1 = -0.1621$ and $b_2 = 0$ a solution of (1-22) is $U = -0.1621X_t$ by (27) . Substituting this into (21) results in $a_1 = 0.0188$ and a_2 =0.003 . From the linear equation

$$dY_t = (a_1(t,Y_t)X_t + a_2(t,Y_t))dt + (b_1(t,Y_t)X_t + b_2(t,Y_t))dW_t)$$

where $\mathrm{X_t} = \mathrm{Y_t}^{-1}$.

The general solution to the SDE of equation (3١), obtained by the reduction method, is

$$X_t = \frac{f_t}{\frac{1}{X_0} + 0.0188\int_0^t f_s ds}$$

where

$$f_t = e^{(0.0057t - 0.1621W_t)}$$

The second model: The nonlinear SDE

$dX_t = \theta_1 X_t^2 dt +$
$\theta_2 X_t dW_t$
(3٢)

Where X_t is the prevalence of HIV, the initial value, $X_0 = 0.10089$, represents the first reading of the diffusion rate for this model. The average prevalence of HIV in the Arab world is given in Table 1 for 33 consecutive

JOBS مجلة العلوم الأساسية
Journal of Basic Science
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العدد الثاني والثلاثون
٢٠٢٥م/١٤٤٧هـ

years. The model parameters (θ_1 and θ_2) will be estimated by the maximum likelihood estimation method . So $N = 33$ and $t = 1$, Equation (33) is the result of estimating the parameters of the SDE for Equation (32), using the maximum likelihood estimation approach .

$$\theta_1 = \frac{\sum_{j=1}^{33}(X_j - X_{j-1})}{\sum_{j=1}^{33}(X_{j-1}^2 \Delta t)} = 0.2519$$

$$\theta_2 = \frac{\sum_{j=1}^{33} X_j - X_{j-1} - \theta_1 X_{j-1}^2 \Delta t}{\sum_{j=1}^{33}(X_{j-1} \Delta t)} = -0.8071 \qquad (33)$$

So the model obtained in this application is

$$dX_t = 0.2519 X_t^2 dt + (-0.8071) X_t dW_t \qquad (34)$$

To find the general solution of Equation (3٤) by the reducible method . where $a(t, X_t) = 0.2519 X_t^2$ and $b(t, y) = (-0.8071) X_t$, so $A \equiv 0.8071 - 0.3121 X_t$. Thus (25) is satisfied with any b_1 . For $b_1 = 0.8071$ and $b_2 = 0$ a solution of (22) is $U = 0.8071 X_t$ by (27) . Substituting this into (21) results in $a_1 = 0.6514$ and $a_2 = -0.2519$. From the linear equation

$$dY_t = (a_1(t, Y_t) X_t + a_2(t, Y_t))dt + (b_1(t, Y_t) X_t + b_2(t, Y_t))dW_t)$$

where $X_t = Y_t^{-1}$.

The general solution to the SDE of equation (3٤), obtained by the reduction method, is

$$X_t = \frac{e^{\{(-0.3257t - 0.8071 w_t\}}}{x_o - 0.2519 \int_o^t e^{\{(-0.3257t - 0.8071 w_t\}} ds}$$

6- Comparison between the proposed models:

The optimal model for representing the time series of the prevalence of HIV in the Arab world was examined by calculating the prediction accuracy of the suggested model.

Table(1): Data on the prevalence of HIV in the Arab world with prediction values the two models resulting from

Year	HIV	Model 1	Model 2	Year	HIV	Model 1	Model 2	Year	HIV	Model 1	Model 2
1990	0.10089	0.10091	0.10089	2001	0.10442	0.10408	0.1019	2012	0.10253	0.10231	0.10291
199	0.1008	0.1007	0.1009	200	0.1043	0.1045	0.1019	201	0.1025	0.1024	0.103

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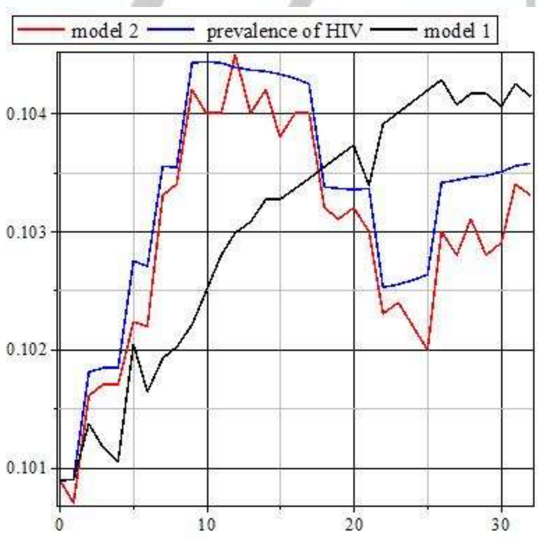
العدد الثاني والثلاثون
٢٠٢٥م /١٤٤٧هـ

1	9	2	8	2	9	1	9	3	5	6	01
199 2	0.1018 1	0.1016	0.1010 7	200 3	0.1043 7	0.1040 3	0.1020 8	201 4	0.1025 9	0.1022 8	0.103 1
199 3	0.1018 4	0.1017 5	0.1011 6	200 4	0.1043 5	0.1042 1	0.1021 7	201 5	0.1026 3	0.1020 7	0.103 19
199 4	0.1018 5	0.1017 1	0.1012 6	200 5	0.1043 3	0.1038 1	0.1022 7	201 6	0.1034 1	0.1030 8	0.103 28
199 5	0.1027 5	0.1022 2	0.1013 5	200 6	0.1042 9	0.1040 2	0.1023 6	201 7	0.1034 4	0.1028 4	0.103 38
199 6	0.1027	0.1022 9	0.1014 4	200 7	0.1042 5	0.1040 4	0.1024 5	201 8	0.1034 6	0.1031 1	0.103 47
199 7	0.1035 6	0.1033 5	0.1015 3	200 8	0.1033 8	0.1032 5	0.1025 4	201 9	0.1034 7	0.1028 4	0.103 56
199 8	0.1035 5	0.1034 7	0.1016 2	200 9	0.1033 6	0.1031 1	0.1026 3	202 0	0.1035 1	0.1029 7	0.103 66
199 9	0.1044 3	0.1042 5	0.1017 1	201 0	0.1033 6	0.1032 4	0.1027 3	202 1	0.1035 5	0.1034	0.103 75
200 0	0.1044 4	0.1040 2	0.1018	201 1	0.1033 6	0.1030 2	0.1028 2	202 2	0.1035 7	0.1033 1	0.103 85

Figure(2): Graph of two models with a data curve



We observe that the first model fits the data better than the second model. We used the mean absolute percentage error (MAPE) to confirm the best model. It is used to measure the accuracy of prediction for different models or for different times. This is calculated by using :

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٢٠٢٥م /١٤٤٧هـ

$$\text{MAPE}=\frac{1}{k}\left(\sum_{j=1}^{k}\frac{|A_j-F_j|}{A_j}\right).100$$

where A_j is the observed value, F_j is the model's predicted value, and k is the number of observations. In (Frechtling , 2012), Table 2 provides interpretations of various MAPE values .

Table (2): Interpretation of MAPE values

Value	Interpretation
MAPE < 0.1	Perfect forecast
0.1 ≤ MAPE < 0.2	Good forecast
0.2 ≤ MAPE < 0.5	Reasonable forecast
MAPE ≥ 0.5	Imperfect forecasting

After finding the absolute relative error rate for the two models (31) and (34), We note that the absolute relative error rate for model (31) is equal to 0.9081, and the absolute relative error rate for model (3٤) is equal to 0.2716. That is, model (34) achieves reasonable forecasts that are better than model (31). That is, the second model is better for modeling data on the prevalence of HIV in the Arab world.

7- Conclusion:

Stochastic differential equations are characterized by the fact that they consist of two parts: the first deterministic term part represents the deterministic term, and the second part represents the stochastic or random part. This stochastic term makes it difficult to know the nature of the behavior of the function over a long period of time or the path taken by the function represented in the form of the SDE. Therefore, most researchers resorted to developing and creating relationships and theories deduced from ordinary differential equations. Before starting the reduction process, it was confirmed whether the nonlinear SDEs were reducible to linear SDEs or not, using some special theories in this field derived from Ito's integral formula. The conversion to linear equations was obtained, through which the analytical solution can be found precisely. In the applied aspect of estimating model parameters, the maximum likelihood estimation method was used. This was done by studying the time series of the prevalence of HIV in the

Arab world, and two models of SDEs were applied. The second model was more efficient than the first model .

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