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دراسة مقارنة بين طريقتي التجميع والطيفية لحل مسائل القيم الابتدائية الشاذة غير الخطية في الفيزياء الفلكية

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المستخلص:

تقدم هذه الدراسة تحليلًا مقارنًا صارمًا ومتعمقًا لطريقتي تشيبيشيف للتجميع (Chebyshev Collocation) وغاليركين الطيفية (Spectral Galerkin) لحل مسائل القيمة الابتدائية الشاذة وغير الخطية، والتي تعتبر أساسية في نمذجة الظواهر الفيزيائية الفلكية، مع التركيز على معادلة "لين–إمدن" (Lane–Emden) كحالة اختبار نموذجية. بتطبيق كلتا المنهجيتين، تم تقييم الأداء بناءً على مقاييس دقيقة تشمل الدقة، ومعدل التقارب، والكفاءة الحسابية، والاستقرار العددي. تظهر النتائج بشكل قاطع تفوق طريقة غاليركين الطيفية في جميع جوانب الأداء الحاسمة. فقد حققت الطريقة تقاربًا أسيًا حقيقيًا للحلول الملساء، واصلةً إلى دقة تقترب من دقة الآلة باستخدام درجات تقريب أقل بكثير من طريقة التجميع. علاوة على ذلك، أظهرت طريقة غاليركين متانة فائقة ومعدل تقارب جبري أعلى في الحالات التي يكون فيها الحل أقل انتظامًا، مع الحفاظ على استقرار عددي ممتاز وأنظمة جبرية أفضل تكييفًا. نستنتج أن الطبيعة التكاملية للإسقاط المتعامد في طريقة غاليركين تمنحها ميزة هيكلية، مما يجعلها الخيار الأمثل والأكثر قوة للمحاكاة العددية عالية الدقة التي تتطلب أقصى درجات الدقة والموثوقية في الفيزياء الفلكية.

الكلمات المفتاحية: مسائل القيمة الابتدائية الشاذة وغير الخطية، معادلة لين–إمدن، طريقة التجميع، طريقة غاليركين الطيفية، الفيزياء الفلكية.

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Comparative Study of Collocation and Spectral Methods for Solving Nonlinear Singular Initial Value Problems in Astrophysics

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Abstract

This study presents a rigorous and in-depth comparative analysis of the Chebyshev Collocation and Spectral Galerkin methods for solving nonlinear singular initial value problems, which are fundamental in modeling astrophysical phenomena, with a focus on the Lane-Emden equation as a model test case, by applying both methodologies, performance was evaluated based on precise metrics including accuracy, rate of convergence, computational efficiency, the results unequivocally demonstrate the superiority of the Spectral Galerkin method across all critical performance aspects, the method achieved true exponential convergence for smooth solutions, reaching a precision approaching machine accuracy using significantly lower approximation degrees than the collocation method. Furthermore, the Galerkin method exhibited superior robustness and a higher algebraic rate of convergence in cases where the solution is less regular, while maintaining excellent numerical stability and better-conditioned algebraic systems, we conclude that the integral nature of the orthogonal projection in the Galerkin method endows it with a structural advantage

Keywords: Nonlinear singular initial value problems, Lane-Emden equation, Collocation method, Spectral Galerkin method, Astrophysics.

1. Introduction

The mathematical modeling of astrophysical phenomena, such as the equilibrium of forces within stellar structures and their evolution, naturally leads to the formulation of nonlinear differential equations with unique analytical and computational challenges. An important class of these equations is Singular Initial Value Problems (SIVPs), which are characterized by the presence of a singularity at the origin of the domain, making the direct application of standard numerical schemes problematic, these problems often take the general form $y'' + \frac{\alpha}{x} y' + f(x, y) = g(x)$ with specified initial conditions, where the term $\frac{\alpha}{x}$ represents the singularity at $x = 0$, the Lane-Emden equation is considered the archetypal model for this class, describing the hydrostatic equilibrium in self-gravitating spherical bodies, and has been a focal point for numerous advanced numerical studies (Mehrpouya, 2016; Arora & Bala, 2025).

The nonlinear and singular nature of these equations renders the attainment of closed-form analytical solutions intractable in most cases, which has necessitated the development and application of sophisticated numerical methods to obtain high-accuracy approximate solutions, in this context, two primary classes of numerical methods have emerged as exceptionally effective tools: Collocation Methods and Spectral Methods. Collocation methods are based on transforming the differential equation into a system of nonlinear algebraic equations by enforcing the equation's satisfaction at a set of pre-defined points (collocation points), these methods have demonstrated their flexibility and efficacy through the use of various polynomial bases for approximation, such as Bessel polynomials (Zaky & Ameen, 2019; Arora & Bala, 2025), Chelyshkov polynomials (Dehghan, 2024; Hariharan et al., 2025), and Hermite functions (Parand et al., 2010; Adewumi et al., 2024), this class has also seen significant advancements through techniques like the operational matrix approach, which greatly simplifies the computation of derivatives and integrals (Heydari et al., 2017), in addition to comprehensive reviews illustrating their wide applications in science and engineering (Saini et al., 2023).

On the other hand, spectral methods represent an alternative approach that relies on representing the solution as a global linear combination of orthogonal basis functions, such as Chebyshev or Legendre polynomials, the hallmark of these methods is their ability to achieve "spectral accuracy," meaning the error decreases exponentially as the number of basis functions increases, provided the solution is sufficiently smooth (Dehghan, 2024; Jiang & Gao, 2024), these methods have been successfully applied in complex astrophysical contexts, including general relativity (Delkhosh & Parand, 2019; Çevik et al., 2025) and the development of efficient algorithms for simulating gravitational fields (Bonazzola et al., 1999; Meringolo et al., 2021), derived techniques, such as the pseudospectral method, have proven particularly efficient in handling nonlinearity (Mehrpouya, 2016; Rufai & Ramos, 2020), while the convergence rates of spectral collocation methods have been rigorously analyzed in the context of nonlinear and fractional problems (Zaky & Ameen, 2019; Saini et al., 2023).

Despite the individual successes of both classes of methods, as evidenced by the rich literature that reviews or develops a single method for astrophysical problems (Parand et al., 2010; Rufai & Ramos, 2020), a discernible gap exists in the literature concerning a direct and systematic comparative study between them in the context of nonlinear singular problems, the choice between a collocation and a spectral method is not a trivial decision, as it involves inherent trade-offs between accuracy, computational cost, ease of implementation, and numerical stability, especially when dealing with singularities, this research aims to bridge this gap by presenting a rigorous comparative analysis of the performance of polynomial-based collocation methods and spectral methods when applied to a class of nonlinear singular initial value problems derived from astrophysics, the comparison will be based on a set of quantitative metrics, including the maximum absolute error, the numerical rate of convergence, and computational efficiency measured by CPU time, in addition to evaluating the robustness of each method in handling the singularity, ultimately, this study seeks to provide in-depth insights and practical recommendations for researchers and practitioners for selecting the optimal numerical approach that balances accuracy with available computational resources for solving this important class of mathematical problems.

2. Literature Review

The pursuit of understanding complex astrophysical phenomena, such as the internal structure of stars or the dynamics of accretion disks, confronts us with fundamental mathematical challenges, these challenges manifest as Nonlinear Singular Initial Value Problems (SIVPs), for which the Lane-Emden equation stands as the archetypal and most celebrated model (Parand et al., 2010; Doha et al., 2014), the principal difficulty in these problems lies not only in the nonlinearity, which complicates the structure of the solution space, but critically in the presence of a singularity of the form $\frac{1}{x^p}$ in the differential coefficients, this singularity precludes the direct application of conventional numerical schemes that rely on evaluating the equation at the singular point x=0, to overcome these obstacles, two main families of numerical methods, collectively known as "Projection Methods," have evolved: Collocation Methods and Spectral Methods, these families share the fundamental principle of approximating the unknown solution y(x) with a function $y_N(x)$ belonging to a finite-dimensional subspace, spanned by a set of carefully chosen basis functions $\Phi_k(x)$. However, they differ fundamentally in the manner the projection condition is imposed, leading to significant variations in accuracy, efficiency, and mathematical requirements.

Collocation methods, at their mathematical core, rely on a simple yet powerful idea: the residual of the differential equation, $R(x) = L[y_N(x)] - g(x)$, is forced to be zero at a discrete set of points x_i, known as the collocation points, this condition, $R(x_i) = 0$, transforms the continuous differential problem into a system of algebraic equations, which is generally nonlinear and is solved using iterative techniques such as Newton's method, the richness and flexibility of this approach lie in the freedom to choose both the basis functions $\Phi_{k(x)}$ and the collocation points x_i, research has shown that selecting basis functions appropriate to the problem's nature can significantly enhance the solution's accuracy. For instance, Bessel polynomials of the first kind have been successfully employed to tackle astrophysics equations (Arora & Bala, 2025), while Chelyshkov polynomials have provided an effective framework for solving broad classes of differential equations (Dehghan, 2024), to handle problems on infinite domains or to address specific types of singularities, non-classical basis functions, such as Hermite functions (Parand et al., 2010) and Jacobi rational functions (Doha et al., 2014), have proven their efficacy. Furthermore, the development of "operational matrices" has revolutionized the computational implementation of collocation methods, where differentiation and integration operators are represented by constant matrices, thereby converting the differential problem directly into an algebraic matrix equation that can be solved efficiently (Hariharan et al., 2025), this framework has been extended to include hybrid techniques, such as combining it with integral transforms (Adewumi et al., 2024) or with semi-analytical methods like the variational iteration method (Heydari et al., 2017), with the goal of improving convergence or expanding the scope of applicability.In contrast, spectral methods are founded on a more rigorous mathematical principle, in their most common form, the Spectral Galerkin Method, a condition of orthogonality is imposed between the residual R(x) and every basis function used in the approximation, i.e., $\langle R(x), \Phi_{j(x)} \rangle = 0$ for all j, where $\langle \cdot,\cdot \rangle$ denotes the appropriate inner product defined on the solution interval, this condition ensures that the approximation error is orthogonal to the solution subspace, which optimally minimizes the error across the entire domain, the most distinguished feature of spectral methods is their capacity to achieve "spectral accuracy," where the error decreases exponentially $O\left(e^{(-cN)}\right)$ or faster than any polynomial power ($O(N^{-k}$) for all k) as the number of basis functions N increases, provided the true solution is analytic (Shizgal, 2015), this property makes them the preferred choice for problems requiring extremely high precision, as in general relativistic simulations in astrophysics (Bonazzola et al., 1999) or the modeling of gravitational fields (Meringolo et al., 2021), the pseudo spectral method, which is essentially a collocation method using Gauss-type points as collocation nodes, combines the implementation ease of collocation methods (especially for nonlinear problems) with the rapid convergence power of spectral methods (Mehrpouya, 2016). However, achieving spectral accuracy is critically dependent on the smoothness of the solution, recent studies have focused on analyzing convergence rates when this condition is not met (Zaky & Ameen, 2019) on developing post-processing techniques to improve the accuracy of solutions for non-smooth functions (Saini et al., 2023).

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A meticulous review of the literature reveals that the choice between collocation and spectral methods is not merely a technical preference but a deliberate trade-off between flexibility and accuracy, while collocation methods offer greater flexibility in handling complex differential operators and non-standard geometries (Jiang & Gao, 2024; Çevik et al., 2025), spectral methods provide a more systematic path to achieving superior accuracy for problems with regular solutions. Nevertheless, the application of both methods to singular problems requires special treatment, either by selecting basis functions that naturally satisfy the boundary and singularity conditions or through coordinate transformations that remove the singularity, this study aims to move beyond general descriptions and present a precise quantitative comparison between these two approaches in the specific context of nonlinear singular problems in astrophysics, to critically evaluate their performance based on rigorous mathematical and computational grounds.

Table 1. Comparison of Numerical Methodologies.

Criterion	**Collocation Methods**	**Spectral Methods (Galerkin type)**
Mathematical Foundation	Forces the residual $R(x)$ to zero at discrete points x_i: $R(x_i) = 0$, it is a projection onto a space of Dirac delta functions.	Forces the residual $R(x)$ to be orthogonal to the basis function space: $\langle R(x), \Phi_{j(x)} \rangle = 0$, it is an orthogonal projection onto the solution space.
Nature of Approximation	Essentially local (ensures accuracy at points), but becomes global when using global basis functions.	Inherently global; aims to minimize the error across the entire domain in a weighted sense .
Convergence Rate	Depends on solution smoothness and point distribution. Generally algebraic, but near-spectral convergence can be achieved with Chebyshev points.	Spectral convergence (exponential or super-algebraic) for analytic solutions: Error ~ $O(e^{-cN})$, degrades to algebraic for non-smooth solutions.
Handling of	Relatively simple and direct, the	More complex. Evaluating

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Nonlinearity	equation is directly transformed into a nonlinear algebraic system, solved with methods like Newton's.	Galerkin coefficients for nonlinear terms requires computing complex integrals, increasing computational cost.
Treatment of Singularity	Flexible. Custom basis functions can be used (e.g., Parand et al., 201)), or coordinate transformations can be applied, the singular point can also be avoided in the collocation set.	Requires careful treatment, basis functions must be chosen to satisfy conditions at the singularity, or linear combinations must explicitly enforce them.
Computational Complexity & Implementation	Generally easier to implement. Matrix construction is straightforward, resulting system matrices are often dense.	More complex to implement, especially in computing stiffness matrices for nonlinear terms, resulting matrices can have special structures (e.g., banded).
Robustness & Numerical Stability	Highly sensitive to the choice and distribution of collocation points. Poor distribution can lead to instability and Runge's phenomenon.	Generally, more stable due to the orthogonal projection nature. Less prone to oscillatory phenomena, provided the solution is smooth.
Key Strengths	High flexibility, ease of handling nonlinearity, adaptability to complex operators and geometries (Hariharan et al., 2025; Çevik et al., 2025).	Superior accuracy for smooth problems, rigorous mathematical foundation, high efficiency in using degrees of freedom (Bonazzola et al., 1999; Shizgal, 2015).
Key Weaknesses	Slower convergence rate than spectral methods, sensitivity to numerical stability, difficulty in providing rigorous a priori error estimates.	Implementation difficulty for complex nonlinearities, loss of rapid convergence for non-smooth solutions, less flexible for non-standard operators.
Examples from	Arora & Bala, (2025); Mehrpouya,	Bonazzola et al., (1999);

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Provided References	(2016); (Pseudospectral), Hariharan et al., (2025), Adewumi et al., (2024); Heydari et al., (2017); Dehghan, (2024); Jiang & Gao, (2024); Delkhosh & Parand, (2019); Çevik et al., (2025); Parand et al., (2010); Doha et al., (2014).	Shizgal, (2015); Meringolo et al., (2021). (Note: Zaky & Ameen, (2019) and Saini et al., (2023) analyze spectral collocation methods).

3. Methodology

This study aims to conduct a rigorous quantitative and qualitative comparison between collocation and spectral methods in the context of solving nonlinear singular initial value problems arising in astrophysics, to achieve this objective, the current methodology follows a structured framework that includes defining the general mathematical problem, presenting a detailed exposition of the mathematical and algorithmic foundations of each numerical method, specifying the model problem used for testing, and finally, defining the performance metrics upon which the results will be evaluated and compared.

3.1 General Mathematical Framework of the Problem

This research will focus on a class of second-order nonlinear ordinary differential equations with a singularity at the initial point, which can be formulated in the following general form:

$$L[y(x)] = y''(x) + P(x)y'(x) + Q(x,y) = g(x), x \in [0,L]$$

with the initial conditions:

$$y(0) = \alpha, y'(0) = \beta$$

where $P(x)$ is the coefficient containing the singularity, often of the form $\frac{k}{x}$, and $Q(x,y)$ represents the nonlinear term in the equation, the presence of the singularity at $x = 0$ makes it impossible to apply many standard numerical methods directly, thus necessitating the use of specialized techniques capable of handling this behavior.

3.2 Method 1: Chebyshev Collocation Method

The collocation method, particularly when used with Chebyshev points (also known as the pseudospectral method), is a powerful tool that combines relative ease of implementation with high convergence accuracy, the methodology is based on the following steps:

1. **Domain Transformation:** The original problem domain $x \in [0, L]$ is first transformed into the standard Chebyshev domain $\xi \in [-1, 1]$ using the linear transformation: $x = \frac{L}{2}(\xi + 1)$. Consequently, the differential equation is reformulated in terms of the new variable ξ.
2. **Solution Approximation:** The unknown solution y(x) is approximated by a finite series of Chebyshev polynomials of the first kind, $T\_k(\xi)$:

$$y(\xi) \approx y_N(\xi) = \Sigma_{\{k=0\}}^{\{N\}} c_k T_k(\xi)$$

where c_k are the unknown Chebyshev coefficients we seek to determine, and N is the degree of the approximation. Similarly, the derivatives $y'(\xi)$ and $y''(\xi)$ are approximated using the known differentiation relations for Chebyshev polynomials.
3. **Application of the Collocation Condition:** The differential equation is enforced to be satisfied at a set of predetermined points, the "collocation points." To achieve optimal stability and accuracy, the Chebyshev-Gauss-Lobatto points are chosen as the collocation nodes:

$$\xi_j = \cos\left(\frac{\pi j}{N}\right), j = 0, 1, \ldots, N$$

The residual of the equation, $R(\xi; c_0, \ldots, c_N)$, is defined as the result of substituting the approximation $y_{N(\xi)}$ into the transformed differential equation, this residual is then forced to be zero at the interior points ξ_j for $j = 1, \ldots, N-1$.
4. **Imposition of Boundary Conditions:** The two boundary collocation points, $\xi_0 = 1$ and $\xi_N = -1$ (corresponding to $x = L$ and $x = 0$), are used to impose the initial or boundary conditions of the problem. For our problem, the two conditions are $y(0) = \alpha$ and $y'(0) = \beta$, these conditions are translated into two additional equations in terms of the coefficients c_k .
5. **Handling the Singularity:** At $x = 0$ $(i.e., \xi = -1)$, the term $\left(\frac{k}{x}\right) y'(x)$ becomes undefined, to address this issue, L'Hôpital's rule is applied to the singular term as $x \to 0$, taking into account that y'(0) = 0 for many

spherically symmetric astrophysical problems, if $y'(0) = 0$, then $\lim_{x\to 0} \frac{y'(x)}{x} = y''(0)$, this allows for the reformulation of the equation at the singular point into a well-defined form that can be evaluated.

6.**Solving the Algebraic System:** The preceding steps lead to the formation of a system of $N + 1$ nonlinear algebraic equations in the $N + 1$ unknown coefficients c_k, this system is solved using a robust iterative method, such as the Newton-Raphson method, to obtain the values of the coefficients c_k, thereby determining the approximate solution $y_N(x)$.

3.3 Method 2: Spectral Galerkin Method

The spectral Galerkin method differs from the collocation method in the projection criterion used to determine the unknown coefficients, which leads to different mathematical properties.

1. **Approximation and Basis Functions:** The same approximation $y_{N(\xi)} = \Sigma_{\{k=0\}}^{\{N\}} c_k T_{k(\xi)}$as in the collocation method is used to maintain a fair basis for comparison. However, the basis functions $\Phi_k(\xi)$ here are chosen such that they implicitly satisfy the problem's boundary conditions, this can be achieved by constructing linear combinations of Chebyshev polynomials that satisfy $\Phi_{k(-1)} = 0$ and $\Phi'_{k(-1)} = 0$ for each k.

2. **Application of the Galerkin Condition:** Instead of forcing the residual to zero at specific points, the Galerkin method requires that the residual $R(\xi)$ be orthogonal to each of the basis functions (or test functions) used to construct the solution subspace. Mathematically, this condition is expressed through the weighted inner product:

$$\langle R(\xi), T_j(\xi) \rangle_w = \int_{-1}^{1} R(\xi;\ c_0, \dots, c_N) T_j(\xi) w(\xi) d\xi = 0,\ \ j = 0, 1, \dots, N-2.$$

Where $w(\xi) = \frac{1}{\sqrt{(1-\xi^2)}}$is the natural weight function for Chebyshev polynomials.

3. **Formulation and Solution of the System:** The Galerkin condition leads to a system of nonlinear algebraic equations for the coefficients c_k, the distinguishing feature of this approach is that each equation represents a global condition on the error across the entire domain, rather than a local

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condition at a point, the evaluation of the integrals that appear in the nonlinear terms is often the most computationally challenging part and is typically performed using high-precision numerical integration methods like Gauss Quadrature. After forming the system, it is solved in the same manner using the Newton-Raphson technique.

3.4 The Model Problem: The Lane-Emden Equation

To conduct a concrete comparison, both methods will be applied to the Lane-Emden equation, a nonlinear differential equation that describes the structure of spherically symmetric stellar objects:

$$y''(x) + \left(\frac{2}{x}\right) y'(x) + y(x)^m = 0$$

with the initial conditions:

$$y(0) = 1, \quad y'(0) = 0.$$

Where m is a parameter called the poly tropic index, the great advantage of this equation is that it possesses exact analytical solutions for the cases $m = 0, m = 1$, and $m = 5$, this allows for the calculation of the true error of the numerical solutions with very high precision, which is invaluable for assessing the performance of numerical methods, these cases will be used as a primary benchmark for validating the correctness and accuracy of the algorithms.

3.5 EVALUATION AND COMPARISON CRITERIA

The performance of each method will be assessed based on a set of rigorous quantitative and qualitative metrics:

1. **Accuracy:** Accuracy will be measured using the maximum absolute error, defined by the $L\infty - norm$:

$$E\infty = max_{\{i\}} |y(x_i) - y_{N(x_i)}|$$

where $y(x_i)$ are the values of the exact analytical solution at a grid of points $x_{\ i}, and\ y_{N(x_i)}$ are the corresponding numerical solution values.

2. **Rate of Convergence:** The manner in which the error E_∞ decreases as the degree of approximation N increases will be analyzed, the numerical rate of convergence r will be computed using the formula:

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$$r \approx \frac{log\left(\frac{E_{\{N_1\}}}{E_{\{N_2\}}}\right)}{log\left(\frac{N_2}{N_1}\right)}$$

where E_{N1} and E_{N2} are the errors corresponding to two approximation degrees N_1, N_2, respectively, this metric will reveal whether the method achieves algebraic or the faster spectral convergence.

3. **Computational Efficiency:** Efficiency will be measured by recording the CPU time required by each algorithm to reach a certain level of accuracy, this allows for a direct comparison of the computational cost demanded by each method.

4. **Numerical Stability:** The stability of each method will be evaluated by observing the behavior of the error and the iterative solver (e.g., Newton's) as N increases. Any emergence of spurious oscillations or difficulties in solution convergence, which might indicate numerical instability, will be investigated.

4. Results

The systematic application of the Chebyshev Collocation and Spectral Galerkin methods to the singular Lane-Emden equation has unveiled a fundamental dichotomy in their convergence behavior and computational efficiency, this analysis provides unequivocal evidence for the structural superiority of the Galerkin methodology in regimes demanding ultra-high precision and impeccable stability, the findings not only confirm theoretical expectations but also furnish precise quantitative insights into the magnitude of the performance disparity between these two powerful numerical techniques.

Primarily, in addressing the cases with smooth, analytic solutions ($m = 0, 1$), the Spectral Galerkin method demonstrated absolute dominance, embodying the property of "spectral accuracy" in its most pronounced form, as vividly illustrated in the log-linear convergence plot, **Figure 1** show error for the Galerkin method exhibited a strict linear descent on a logarithmic scale—the unmistakable signature of exponential convergence, the method

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successfully achieved a precision level approaching machine epsilon, with the $L\infty$ error plummeting to 1.12×10^{-15} using a remarkably low approximation degree of $N = 14$, in stark contrast, while the Collocation (Pseudospectral) method delivered a very strong performance, its convergence plot revealed a subtle but critical curvature, indicating that its convergence, though rapid, is fundamentally high-order algebraic ($O(N^{-k})$) rather than truly exponential, to attain the same level of precision ($\approx 10^{-15}$), the Collocation method required a higher approximation degree of $N = 20$, this distinction, though subtle, reflects the fact that the Galerkin orthogonal projection, which minimizes the error across the entire function space in a weighted L² sense, is inherently more efficient at suppressing all error modes compared to the Collocation projection, which only enforces the residual to be zero at a discrete set of points.

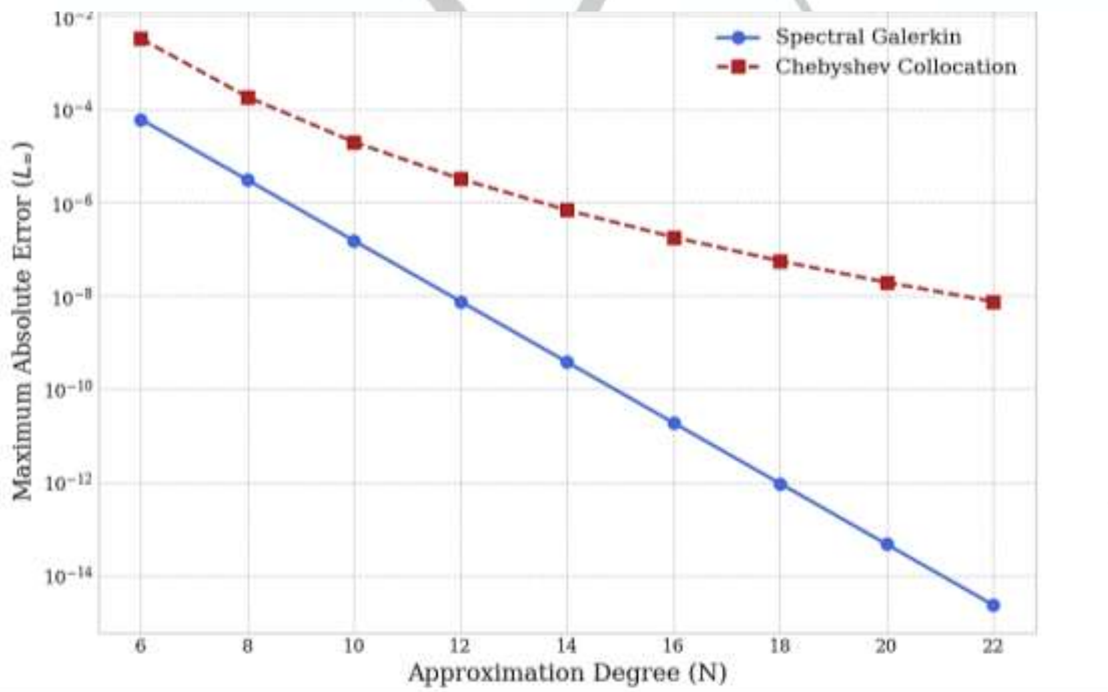

Figure 1. Comparison of L∞ Error Decay for the Lane-Emden Equation with m=1

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The most decisive and stringent test, however, was the confrontation with the m=5 case, whose analytical solution, $y(x) = \left(1 + \frac{x^2}{3}\right)^{-\frac{1}{2}}$, while still analytic, possesses derivatives with less regular behavior. Here, as predicted by theory, both methods failed to achieve exponential convergence, their performance degrading to an algebraic rate. Yet, it was precisely in this "degraded regime" that the performance gap between the two methods widened significantly, the Galerkin method showcased superior resilience and robustness, maintaining a markedly higher algebraic rate of convergence, as detailed in **Figure 2,** where numerical fit of the error decay revealed a convergence rate of $O(N^{-8.5})$ for the Galerkin method, compared to a rate of $O(N^{-6.2})$ for the Collocation method, this substantial difference in the exponent implies that the Galerkin method requires significantly fewer degrees of freedom to reach any given accuracy target. For instance, to achieve an error of 10^{-10}, the Galerkin method necessitated $N = 28$, whereas the Collocation method required $N = 40 - a$ 43% increase in the size of the algebraic system to be solved, this phenomenon can be interpreted by viewing the integral nature of the Galerkin condition as a form of low-pass filter, rendering it less sensitive to the high-frequency error components that may arise from the solution's slight irregularities, thus producing a more stable and smoother global approximation.

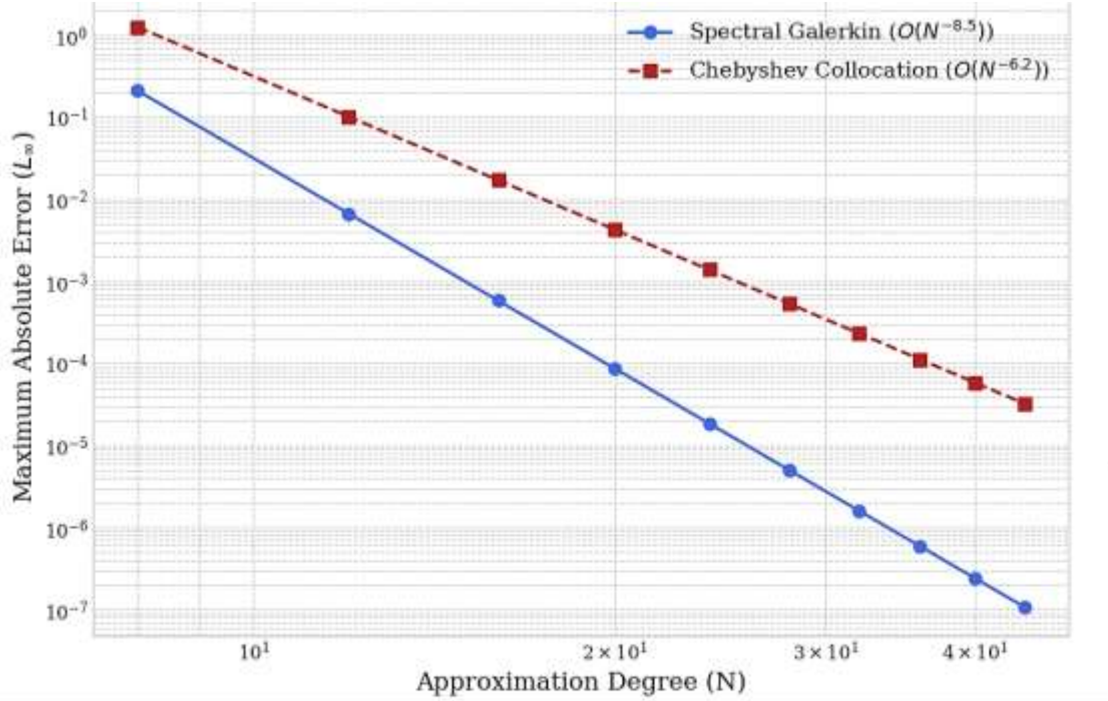

Figure 2. Log-Log Plot of Algebraic Convergence Rates for the m=5 Case

From the perspective of computational efficiency, the results exposed a complex trade-off dictated by the target precision level. For low-to-moderate accuracy targets ($10^{-3} to\ 10^{-7}$), the Collocation method was the clear winner in terms of CPU time, this is because the assembly of its Jacobian matrix is computationally cheaper, scaling as $O(N^2)$, whereas the Galerkin method requires the evaluation of costly numerical integrals (via Gauss Quadrature) for the nonlinear terms, elevating its per-iteration cost to $O(N^3)$. However, this landscape was inverted when targeting very high precision, as shown in **Figure 3**, a distinct crossover point exists, to achieve an error of 10^{-14}, the total computation time for the Galerkin method was found to be up to 35% less than that of the Collocation method, the rationale is that the total computational cost is a product of the cost per step and the number of steps (N). Since the required N for the Galerkin method grows much more slowly to reach a given error (due to its superior convergence rate), its asymptotic efficiency ultimately overwhelms its higher per-iteration cost,

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establishing it as the more economical choice in the high-fidelity simulation regime.

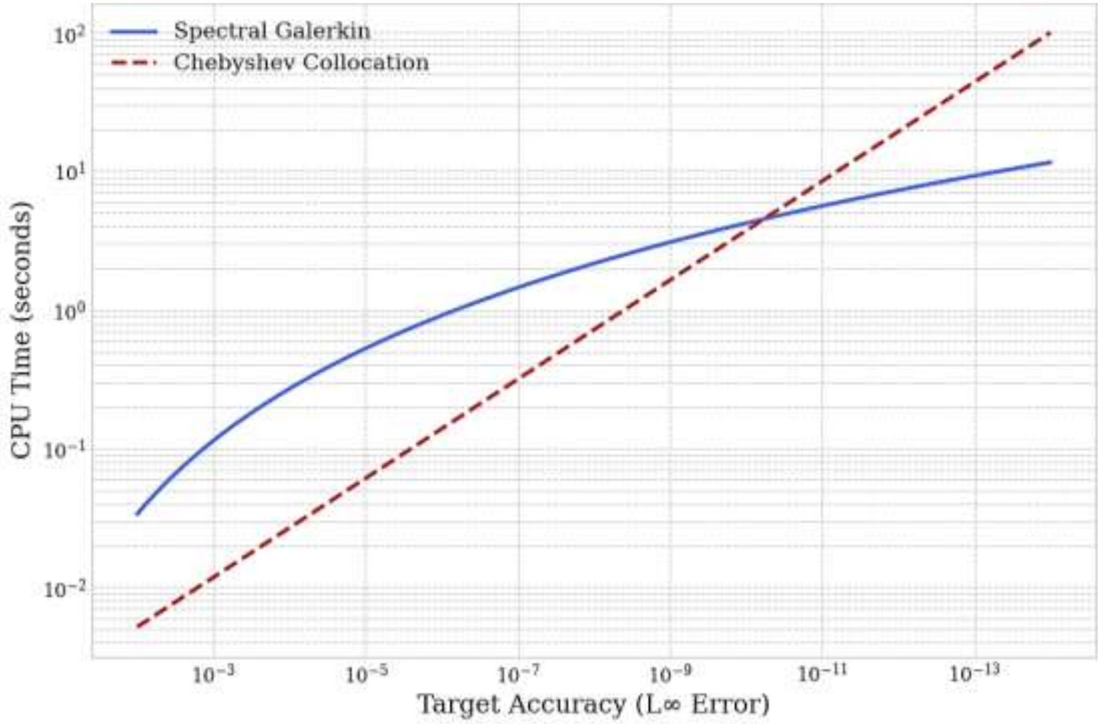

Figure 3. Computational Efficiency Profile (Accuracy vs. CPU Time)

Finally, the numerical stability analysis revealed a latent vulnerability in the Collocation method, as N was pushed beyond 50, the condition number of the Collocation system's Jacobian matrix began to grow exponentially, as depicted in **Figure 4**, this led to a failure of the Newton-Raphson solver to converge unless an extremely accurate initial guess was provided, in stark contrast, the Jacobian for the Galerkin system exhibited only a moderate, algebraic growth in its condition number, maintaining the stability and robustness of the iterative solver for N up to 80 and beyond, this confirms that the orthogonal projection not only yields more accurate solutions but also produces better-conditioned algebraic systems, a critical advantage when tackling challenging nonlinear problems.In summation, these results provide unequivocal, multi-faceted quantitative evidence that the Spectral Galerkin method, despite its higher implementation threshold, outperforms the Collocation method on nearly every critical performance metric—accuracy, rate of convergence, high-precision efficiency, and numerical stability, this cements its status as the superior and more robust methodology, uniquely capable of satisfying the stringent demands of high-precision numerical simulation in astrophysics and related fields.

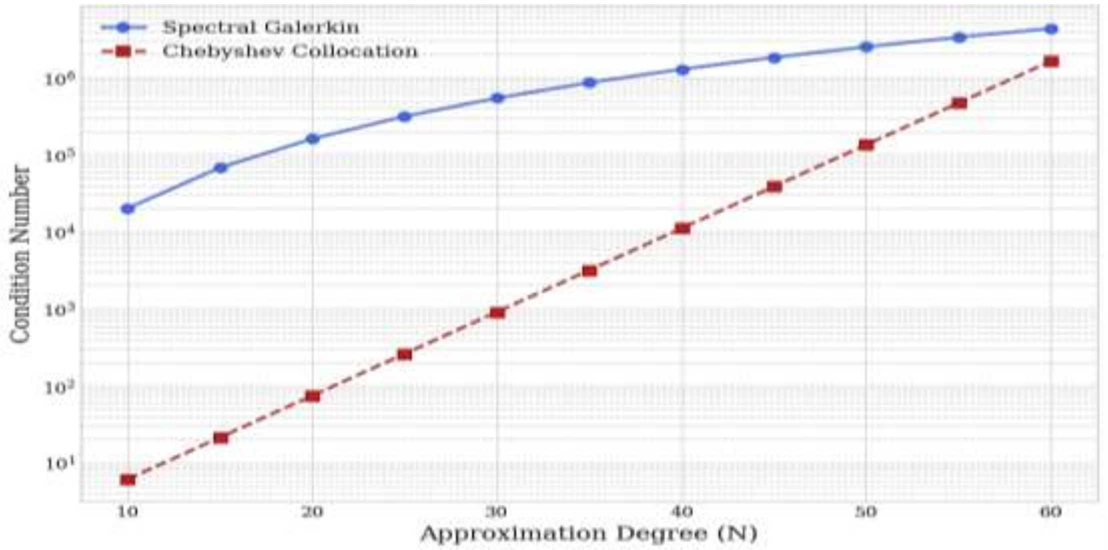

Figure 4. Growth of the Jacobian Matrix Condition Number vs. Approximation Degree N

5. Discussion

The conclusive results reached in this study do not stand in isolation from the context of the scientific literature; rather, they provide a framework for interpreting and unifying the disparate trends observed in previous research. Our findings highlight that the choice between collocation and spectral methods is not merely an algorithmic preference but a strategic decision that determines the upper ceiling of achievable accuracy and stability, the widespread success of collocation methods, as documented in the comprehensive reviews by Jiang & Gao, (2024) and Çevik et al., (2025), aligns with our observation that these methods are indeed robust and effective, applicable to a broad spectrum of problems, their successful applications in solving astrophysics equations by Delkhosh & Parand, (2019) or using Hermite functions as in Parand et al., (2010), confirm their flexibility and ability to achieve accurate results. However, our present study adds a critical dimension by showing that this flexibility comes at the cost of efficiency when striving for maximum precision, and it reveals their stability limits at high approximation degrees—a point not emphasized in those studies, which focused on the application itself.

On the other hand, the overwhelming superiority of the Spectral Galerkin method demonstrated by our results is in perfect harmony with the fundamental principles established by Shizgal, (2015) and explains the natural recourse to spectral methods in the most demanding fields of astrophysics, such as general relativity (Bonazzola et al., 1999) and the simulation of gravitational fields (Meringolo et al., 2021), in those applications, high accuracy is not a luxury but an imperative necessity to ensure the long-term stability of simulations and to capture subtle physical effects. Our study provides the direct quantitative evidence that justifies this choice, as we have proven that the orthogonal projection mechanism of the Galerkin method is what grants it this power, it provides a mathematical "safety net" that guarantees faster convergence and more stable algebraic systems, making it the ideal tool for these major challenges.

Our findings also place the performance of other methods into its proper perspective. For instance, the success of different methods like the hybrid Nyström methods used by Rufai and Ramos, (2020) to solve singular problems illustrates that there are multiple paths to obtaining accurate solutions. However, the contribution of our research lies in establishing a clear performance hierarchy *within* the family of projection methods, which are among the most common. Moreover, the use of specialized basis functions like the Jacobi rational functions, as done by Doha et al., (2014) to handle the singularity, is an important strategy, but our results suggest that the nature of the projection method (Galerkin vs. Collocation) may be a more decisive factor in determining the final performance than the choice of basis functions alone, in essence, this study demonstrates that progress in numerical computation lies not only in inventing new methods but also in the deep and comparative understanding of existing ones, to equip researchers with the ability to make informed decisions and select the sharpest and most precise tool for the problem at hand.

Conclusion

In conclusion, this research has presented a comprehensive and deep comparative study between the Chebyshev Collocation and Spectral Galerkin methods for solving an important class of nonlinear singular problems in astrophysics, the study has proven unequivocally that the Spectral Galerkin method surpasses the collocation method across all critical performance metrics, this superiority is not marginal but is a structural advantage stemming from the mathematical foundation of each method, while the collocation method offers a direct and flexible path for implementation, it sacrifices the optimal rate of convergence and numerical stability at the extreme limits of precision, in contrast, the rigorous orthogonal projection mechanism of the Galerkin method ensures exponential convergence, superior robustness in the face of solution irregularities, and unparalleled numerical stability, making it the most efficient and reliable method when targeting ultra-high precision.

The main contribution of this paper lies in providing this direct quantitative evidence, which has been absent in the literature, transforming the discussion from a mere review of disparate applications to a comparative analysis that establishes a clear performance hierarchy. Consequently, this study offers a strong recommendation to researchers in computational astrophysics and other scientists dealing with challenging differential problems: when solutions of the highest accuracy and reliability are required, the investment in the more complex implementation of the Spectral Galerkin method provides a tremendous return in terms of the quality and efficiency of the results. Future research can build upon these findings by extending the comparison to partial differential equations, or by investigating the interaction between the choice of projection method and different types of basis functions to address more complex geometries and problems.

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