JOBS مجلة العلوم الأساسية
Journal of Basic Science
Print -ISSN 2306-5249
Online-ISSN 2791-3279
العدد الثالث والأربعون
٢٠٢٦م /١٤٤٧هـ
العدد الثالث والأربعون
(٤٥٧)- (٤٩٦)

الاختيار التكيفي للانتظامية عبر الاستقرار في الانحدار متعدد المتغيرات

ضمانات نظرية والتحقق في التطبيقات الواقعية

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المستخلص:

يتطلب الانحدار متعدد المتغيرات عالي الأبعاد استخدام أساليب الانتظامية من أجل تحقيق تقدير مستقر واختيار فعّال للمتغيرات، غير أن تحديد معامل الضبط الذي يفضي إلى استرجاع نموذجي قابل للتكرار ومتسق ما يزال يمثل تحديًا مفتوحًا. في هذا البحث، نقترح طريقة **الاختيار التكيفي للانتظامية عبر الاستقرار** - (*Adaptive Regularization Selection via Stability* *ARSS*)، وهي إجراء ذو مرحلتين يقوم على :

١. بناء أوزان تكيفية مستندة إلى البيانات لعقوبة محدبة موزونة من نوع l_1 ،

٢. واختيار مستوى الانتظامية من خلال تصغير مؤشر عدم استقرار قائم على إعادة المعاينة الجزئية (*subsampling*) ومحسوب عبر شبكة من قيم λ.

نطوّر إطارًا نظريًا موحدًا يثبت وجود المقدِّر ووحدانيته، وحدود الاستقرار في العينات المحدودة (الاضطراب/ليبشيتز)، واتساق الاختيار؛ إذ إن قيمة الانتظامية المختارة بالاستقرار λ^* تحقق معدل التدرج عالي الأبعاد :

$$\lambda^* \asymp \sigma\sqrt{\frac{\log p}{n}}$$

وتؤدي إلى :

$$\Pr(\hat{S}_{\lambda^*} = S_0) \to 1$$

في ظل الشروط المعيارية الخاصة بالقيمة الذاتية المقيّدة (*restricted eigenvalue*) وشرط الحد الأدنى لمعاملات الانحدار. (*beta-min conditions*) وتحت افتراضات إضافية تتعلق بتناقص الأوزان، يتم إثبات خاصية الأوراكل(*oracle property*) ، بما في ذلك الطبيعة التقاربية على المجموعة الفاعلة.

وعلى المستوى التجريبي، تُظهر محاكاة موسعة عبر قيم مختلفة لكل من n وp، ومستويات التخلخل، وبنى الارتباط، أن طريقة **ARSS** تحقق معدلات أعلى للإيجابيات الحقيقية، ومعدلات أقل بصورة ملحوظة للاكتشافات الخاطئة، واستقرارًا أفضل بكثير في الاختيار مقارنة بكل من : **LASSO**، و**Adaptive LASSO**، و**Elastic Net**، و**SCAD**، وطرائق الاختيار بالاستقرار التقليدية. كما تؤكد تطبيقات التحقق الواقعي على مجموعات بيانات في الطب الحيوي والتنبؤ بالأعطال (مثل بيانات **UCI** القلبية الوعائية وبيانات **NASA C-MAPSS**) قابلية الطريقة للتفسير ومتانتها.

توفّر طريقة **ARSS** مسارًا منهجيًا وقابلًا لإعادة الإنتاج لضبط الانتظامية في مشكلات الانحدار متعدد المتغيرات عالي الأبعاد، مع تقديم توصيات عملية تتعلق بإعادة المعاينة الجزئية وتصميم شبكة قيم λ.

الكلمات المفتاحية: الاختيار بالاستقرار؛ الانتظامية التكيفية؛ الانحدار عالي الأبعاد؛ اتساق الاختيار؛ إعادة المعاينة الجزئية.

ADAPTIVE REGULARIZATION SELECTION VIA STABILITY FOR MULTIVARIATE REGRESSION: THEORETICAL GUARANTEES AND REAL-WORLD VALIDATION

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Abstract

High-dimensional multivariate regression requires regularization for stable estimation and variable selection, yet choosing the tuning parameter that yields reproducible and consistent model recovery remains an open challenge. We propose *Adaptive Regularization Selection via Stability* (ARSS), a two-stage procedure that (i) constructs data-driven adaptive weights for a convex weighted ℓ_1 penalty, and (ii) selects the regularization level by minimizing a subsampling-based instability index computed across a grid of λ. We develop a unified theoretical framework proving existence and uniqueness of the estimator, finite-sample stability (perturbation/Lipschitz) bounds, and selection consistency: the stability-selected λ^* attains the high-dimensional scaling $\lambda^* \asymp \sigma\sqrt{\log p/n}$ and yields $\Pr(\hat{S}_{\lambda^*} = S_0) \to 1$ under standard restricted eigenvalue and beta-min conditions. Under additional weight-decay assumptions an oracle property (asymptotic normality on the active set) is established. Empirically, extensive simulations acr, p, sparsity,p, sparsity and correlation structures show that ARSS attains higher true positive rates, markedly lower false discovery rates, and substantially improved selection stability compared to LASSO, Adaptive LASSO, Elastic Net, SCAD, and classical stability selection. Real-world validations on biomedical and prognostics datasets (e.g., UCI cardiovascular and NASA C-MAPSS) corroborate the method's interpretability and robustness. ARSS provides a principled, reproducible route to tuning regularization in high-dimensional multivariate problems, with practical recommendations for subsampling and grid design.

Keywords: stability selection; adaptive regularization; high-dimensional regression; selection consistency; subsampling.

1 INTRODUCTION

Multivariate regression in high dimensions has become a basic part of statistical modeling today, where the number of predictors, p, can be similar to or larger than the sample size, n. In these conditions classical least squares estimation becomes ill-posed or unstable and regularization methods which impose some form of structural constraints like sparsity are highly sought after (Tibshirani, 1996)

Suppose the following linear regression model

$$Y = X\beta^* + \varepsilon,$$

Y and X are $\mathbb{R}^n$ and $\mathbb{R}^{n\times p}$ valued respectively, and β^* is presupposed sparse. A generalized type of regularized estimators may be characterized as

$$\hat{\beta}_\lambda = \arg\min_{\beta\in\mathbb{R}^p}\{L_n(\beta) + \lambda P(\beta)\},$$

$L_n(\beta)$ is the empirical loss, and $P(\beta)$ is a sparsity penalty.

The LASSO is a popular regularization approach that is both estimative and variable-selective. Nonetheless, the consistency in the selection of models requires restrictive conditions of design (Chetverikov et al., 2021). Adaptive penalties were added to eliminate some of these limitations and to obtain oracle properties on appropriate assumptions (Basu et al., 2024).

Although these breakthroughs have been made, high-dimensional regression still faces two major issues. To begin with, the error in estimation and support recovery is highly sensitive to the choice of the tuning parameter eter λ. Conventional approaches like cross-validation, information criteria are mostly aimed at predictive risks, and do not provide selection consistency and reproducibility Second, variable selection processes are quite sensitive to small changes in the data and hence do not reproduc.

Stability-based procedures have been commonly used to measure the fluctuation of selected models through subsampling. Let $\pi_\lambda(j)$ be the probability of selection of variable j under repeated subsampling. The measure of instability can be defined as

$$\text{Instability}(\lambda) = \sum_{j=1}^{p} \pi_\lambda(j)(1 - \pi_\lambda(j)),$$

which characterises the heterogeneity of the support recovery in subsamples (Kwon et al., 2023) Even though stability selection has been empirically effective, no single theoretical model has been developed to connect stability-based tuning-parameter selection with asymptotic selection consistency in multivariate regression.

Research Gap.

The existing literature either analyses consistency under fixed tuning parameters or offers empirical heuristics for stability without providing an asymptotic justification. No thorough theoretical study has been carried out to verify that stability-based tuning ensures selection consistency in high-dimensional multivariate regression.

Contributions.

This gap is filled in this paper because:

1. Suggesting a regularization procedure with the stability based tuning selection.

2. Creating the appropriate conditions of similarity in selection:

$$\Pr\left(\hat{S}_{\lambda^*} = S_0\right) \to 1 \quad \text{as} n \to \infty.$$

3. Calculating robustness between sampling perturbations.
4. Simulating and supporting theoretical results with empirical data.

2 LITERATURE REVIEW

2.1 Regularized Multivariate Regression

Regularization methods are essential tools for dealing with high-dimensional regression problems. Classical techniques encompass Ridge regression that has an ℓ_2 penalty, LASSO that incorporates ℓ_1 penalty on sparsity and Elastic Net that implements a combination of two penalties (ℓ_2 and ℓ_1) to deal with multicollinearity and grouped variables.

Remedial punishment was created to enhance the consistency of selection and minimize bias. As an illustration, to prove that the Adaptive LASSO satisfies properties of oracles, there is a set of appropriate regularity conditions under which data-driven weights on coefficients are computed (Su et al., 2021) Regularized structural equation modeling (SEM) has also been used in structural modeling applications together with stability selection to minimize false positive rate and enhance the reliability of selections (Basu et al., 2024)

Reduced-rank regression (RRR) is a generalization of classical regression for multivariate response-level regression, with a rank constraint on the coefficient matrix. Even though there exist the consistency results of some estimators, in practice AIC, BIC, or cross-validation is frequently used in determining ranks without much theoretical assurances (Chen et al., 2022).

2.2 Stability Selection Methods

The concept of stability selection came up as an overarching paradigm of enhancing the estimation of structures in the high-dimensional context by means of repetitive subsampling (Bodinier et al., 2023) Allow $\pi_\lambda(j)$ to represent the probability of variable j being selected under subsampling. Measures of instability can be obtained as

$$\text{Instability}(\lambda) = \sum_{j=1}^{p} \pi_\lambda(j)(1 - \pi_\lambda(j)).$$

The technique offers error control in finite numbers and enhances replicate.

The stability-based tuning has stability-based extensions such as the StARS algorithm on graphical models which uses the minimal regularization level to obtain a sparse and replicable structure (Křížek et al., n.d.). Mild conditions yield partial sparsistency on theoretical analysis.

Stability-based tuning has also been more recently created with reduced-rank regression (StARS-RRR), whose rank estimation consistency was proved under a stability-driven selection rule (Kalousis et al., n.d.) Tuning parameter selection in penalized regression in general can obtain robust selection consistency under both fixed and diverging dimensions when the tuning parameter is selected by selecting the variables (Kwon et al., 2023).

2.3 Model Selection Consistency in High Dimensions

Several studies have examined the consistency of model selection for penalized estimators. In the case of LASSO-type estimators, inconsistency is conditional on some design conditions, like the irrepresentable condition (Mei & Shi, 2024) More broadly, high-dimensional M-estimator theory is based on the assumption of the restricted eigenvalue or the restricted strong convexity that ensure the convergence and facilitate recovery (Mei & Shi, 2024)It is possible to have adaptive penalties that satisfy oracle properties that is, estimator correctly identifies the true support, and attains the overall asymptotic normality at the active set (Basu et al., 2024). Nevertheless, such results usually presuppose a fixed tuning parameter, or one chosen deterministically as a problem parameter, rather than one chosen for stability.

2.4 Research Gap Summary

Existing literature does not really focus on the relationship between regularization theory and stability-based tuning. Stability selection offers empirical gains and control of finite sampling errors (Pfister et al., 2021) whereas high-dimensional asymptotic theory is used to initiate consistency of penalized estimators under fixed tuning regimes (Yoshikawa & Kawano, 2023). Even though recent studies have identified selection or rank consistency for a given stability-based procedure, there is still no consensus on the theoretical foundation that determines the selection consistency of an adaptive model in regularized multivariate regression, with the tuning parameter selected via a stability-related criterion.

This disparity stimulates the development of a stern, stability-based adaptive regularization structure with demonstrable selection integrity in high-dimensional multivariate regression.

3 PROBLEM FORMULATION

3.1 Model Setup

Consider the high-dimensional linear regression model

$$Y = X\beta^{*} + \varepsilon, \tag{1}$$

where:

- $Y \in \mathbb{R}^{n}$ is the response vector,
- $X \in \mathbb{R}^{n \times p}$ is the design matrix,
- $p \gg n$ (high-dimensional regime),

- $\beta^* \in \mathbb{R}^p$ is the true but unknown parameter vector,
- $\varepsilon \in \mathbb{R}^n$ is a noise vector with $\mathbb{E}(\varepsilon) = 0$ and finite variance.

We assume that β^* is sparse, i.e., only a small subset of its components are nonzero. The true support is defined as

$$S_0 = \{j \in \{1, \dots, p\}: \beta_j^* \neq 0\}. \tag{2}$$

Let $s_0 = |S_0|$ denote the sparsity level.

3.2 Regularized Estimator

To estimate β^* in the high-dimensional setting, we consider a regularized M-estimator of the form

$$\hat{\beta}_\lambda = \arg\min_{\beta \in \mathbb{R}^p} \{L_n(\beta) + \lambda P(\beta)\}, \tag{3}$$

where:

- $L_n(\beta)$ is a convex empirical loss function (e.g., least squares),
- $P(\beta)$ is a sparsity-inducing penalty,
- $\lambda > 0$ is the regularization (tuning) parameter.

The estimated support is defined as

$$\hat{S}_\lambda = \{j: \hat{\beta}_{\lambda,j} \neq 0\}. \tag{4}$$

3.3 Stability Measure

To quantify the reproducibility of variable selection, we adopt a subsampling-based stability framework. Let $\pi_\lambda(j)$ denote the selection probability of variable j under repeated subsampling:

$$\pi_\lambda(j) = \mathbb{P}(j \in \hat{S}_\lambda^{(b)}),$$

where $\hat{S}_\lambda^{(b)}$ is the selected support from the b-th subsample.

We define the instability index as

$$\mathrm{Instability}(\lambda) = \sum_{j=1}^{p} \pi_\lambda(j)(1 - \pi_\lambda(j)). \tag{5}$$

Equivalently, a stability index can be expressed as

$$\mathrm{Stability}(\lambda) = \mathbb{E}[\pi_\lambda(j)(1 - \pi_\lambda(j))]. \tag{6}$$

Small values of Instability(λ) indicate that the selected model is stable under sampling perturbations. In this work, the tuning parameter λ will be selected according to a stability-driven criterion.

4 PROPOSED STABILITY-BASED ADAPTIVE REGULARIZATION

4.1 Adaptive Weighted Penalty

We consider an adaptive weighted ℓ_1 penalty of the form

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Journal of Basic Science
Print -ISSN 2306-5249
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$$P(\beta) = \sum_{j=1}^{p} w_j\,|\beta_j|, \tag{7}$$

where the weights $w_j > 0$ are constructed from an initial estimator and (optionally) the empirical design covariance. A common and effective choice is

$$w_j = f(\hat{\beta}_j^{\text{init}},\, \hat{\Sigma}_X) = \frac{1}{|\hat{\beta}_j^{\text{init}}|^{\gamma}+\delta} \cdot g_j(\hat{\Sigma}_X) \tag{8}$$

with tuning constants $\gamma > 0$, $\delta > 0$ to avoid division by zero, and a correlation adjustment factor $g_j(\hat{\Sigma}_X)$ (for example $g_j(\hat{\Sigma}_X) = 1$ for plain adaptive LASSO or $g_j(\hat{\Sigma}_X)$ reflecting the jth diagonal element of an eigen-based scaling) to mitigate multicollinearity effects. The initial estimator $\hat{\beta}^{\text{init}}$ can OLS (if p < n$p < n$), ridge, or a preliminary LASSO fit on the full data.

With this penalty, the adaptive regularized estimator becomes

$$\hat{\beta}_\lambda = \arg\min_{\beta\in\mathbb{R}^p}\{L_n(\beta) + \lambda \textstyle\sum_{j=1}^{p} w_j|\beta_j|\}. \tag{9}$$

4.2 Stability-Driven Tuning Selection

We select the tuning parameter, *the starting value of lambda*, by minimizing an instability criterion computed via subsampling. Let $\mathcal{L} = \{\lambda_1, \dots, \lambda_L\}$ be a prespecified grid (typically on the log scale). For each $\lambda \in \mathcal{L}$ and each subsample index $b = 1, \dots, B$, compute the estimator $\hat{\beta}_\lambda^{(b)}$ on the b-th subsample and let $\hat{S}_\lambda^{(b)} = \{j : \hat{\beta}_{\lambda,j}^{(b)} \neq 0\}$. Define the empirical selection probability

$$\hat{\pi}_\lambda(j) = \frac{1}{B}\sum_{b=1}^{B} \mathbf{1}\{j \in \hat{S}_\lambda^{(b)}\}.$$

The instability (empirical) is then

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$$\widehat{\text{Instability}}(\lambda) = \sum_{j=1}^{p} \hat{\pi}_\lambda(j)(1-\hat{\pi}_\lambda(j)).$$

We select

$$\lambda^* = \underset{\lambda \in \mathcal{L}}{\operatorname{argmin}} \widehat{\text{Instability}}(\lambda), \tag{10}$$

optionally subject to a sparsity constraint (e.g. enforce $|\hat{S}_\lambda| \le R$ for some budget R) or using the smallest λ attaining instability below a prespecified threshold τ (StARS-style rule).

4.3 Algorithm

Algorithm 1 (Stability-Based Adaptive Regularization).

1. **Initial weights.** Compute an initial estimator $\hat{\beta}^{\text{init}}$ (e.g. ridge or LASSO on full data) and form weights $w_j = f(\hat{\beta}_j^{\text{init}}, \hat{\Sigma}_X)$ as in (8).

2. **Grid and subsampling.** Choose grid $\mathcal{L} = \{\lambda_1 > \cdots > \lambda_L\}$, number of subsamples B, and subsample size n_{sub} (typical choice $n_{sub} = \lfloor \alpha n \rfloor$ with $\alpha \in (0.5, 0.9)$).

3. **Subsample fits.** For each $b = 1, \ldots, B$:

   (a) Draw a subsample index set $\mathcal{I}_b$ of size n_{sub} (without replacement).

   (b) For each $\lambda \in \mathcal{L}$ compute $\hat{\beta}_\lambda^{(b)}$ by solving (9) restricted to rows in $\mathcal{I}_b$.

   (c) Record selected supports $\hat{S}_\lambda^{(b)}$.

4. **Compute selection probabilities.** For each $\lambda \in \mathcal{L}$ and $j = 1, \ldots, p$ compute $\hat{\pi}_\lambda(j)$ and $\widehat{\text{Instability}}(\lambda)$.

5. **Select $\boldsymbol{\lambda}^*$.** Choose λ^* according to (10) or the threshold rule $\min\{\lambda : \widehat{\text{Instability}}(\lambda) \le \tau\}$.

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6. **Refit final model.** Refit (9) on the full data with λ^* to obtain the final estimator $\hat{\beta}_{\lambda^*}$ and report $\hat{S}_{\lambda^*}$.

4.4 Computational Complexity and Convergence

Time complexity.

Let $T_{\text{fit}}(n', p)$ denote the computational cost of fitting the adaptive regularized estimator on n' samples and p predictors (this depends on the optimization method: coordinate descent, proximal gradient, etc.). The dominant cost of the stability selection procedure is the repeated fitting over B subsamples and L tuning values, hence the total cost is approximately

$$T_{\text{total}} \approx B \cdot L \cdot T_{\text{fit}}(n_{sub}, p) \; + \; T_{\text{fit}}(n, p),$$

where the last term corresponds to the final refit on the full data. For common solvers (coordinate descent for ℓ_1 penalties), $T_{\text{fit}}(n', p) = \mathcal{O}(K\, n'p)$ where K is the number of coordinate-descent passes until convergence; thus

$$T_{\text{total}} = \mathcal{O}(BLKn_{sub}p) + \mathcal{O}(Knp).$$

In practice, one can reduce cost by parallelizing the start-equation B subsample fits and by warm-starting along the start-equationg the λ grid.

Convergence and solver remarks.

Assuming that $L_n(\beta)$ is convex and the penalty is convex (weighted ℓ_1), the problem in (9) is convex and any contemporary convex optimizer (coordinate descent, proximal gradient, FISTA) is Star-convergent to a global minimum. Methods based on coordinate descent (as used by glmnet and others) are generally characterized by linear or superlinear empirical convergence on sparse recovered low-energy solutions; proximal accelerated optimization methods, when suitable Lipschitz gradient assumptions are satisfied, converge to the optimal solution at rate proportional to $1/t^2$ (the square of the iteration number). In the case of nonconvex penalties (e.g.,

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SCAD), there is no assurance of convergence to a global optimum, and local convergence optima must be used; our theoretical analysis is based on convex weighted $\ell 1$ penalties, though the algorithm can be empirically adapted to nonconvex penalties.

Practical choices.

Recommended default choices that balance accuracy and cost are:

- $B \in [50,200]$ (larger B yields more stable probability estimates but higher cost).
- Log-spaced grid $\mathcal{L}$ with $L \in [50,100]$ values.
- Subsample fraction $\alpha \in [0.6,0.8]$ (smaller α increases variability of subsamples and may improve sensitivity to unstable variables).
- Warm starts across λ and parallel computation across subsamples to reduce wall-clock time.

The above procedure yields an adaptive, stability-aware tuning selection and a final estimator tailored for reproducible support recovery in high-dimensional multivariate regression.

5 THEORETICAL PROPERTIES

This section states the main assumptions and theorems. Proof sketches are provided; full proofs are deferred to the Appendix.

5.1 Assumptions

We list standard high-dimensional assumptions used in subsequent theorems.

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Journal of Basic Science
Print -ISSN 2306-5249
Online-ISSN 2791-3279
العدد الثالث والأربعون
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• **Sparsity.** The true parameter $\beta^* \in \mathbb{R}^p$ has support S_0 with $s_0 = |S_0|$ and $s_0 = o(n)$ (more precisely $s_0 \log p = o(n)$ under the rates below).

• **Design: Restricted Eigenvalue (RE) condition.** There exists a constant $\kappa_{\min} > 0$ such that for all $\Delta \in \mathbb{R}^p$ with $\| \Delta_{S_0^c} \|_1 \leq 3 \| \Delta_{S_0} \|_1$,

$$\frac{1}{n} \| X\Delta \|_2^2 \geq \kappa_{\min} \| \Delta \|_2^2.$$

This is the usual $\mathrm{RE}(s_0, 3)$ condition.

• **Noise: sub-Gaussian tails.** The entries of the noise vector ε are independent, mean-zero and sub-Gaussian with parameter σ^2; i.e. for all $t \in \mathbb{R}$ and any fixed $u \in \mathbb{R}^n$,

$$\mathbb{E}\exp(t\, u^\top \varepsilon) \leq \exp\left(\frac{t^2 \sigma^2 \|u\|_2^2}{2}\right).$$

• **Weights regularity.** The adaptive weights w_j used in (7) satisfy: there exist constants $0 < w_{\min} \leq w_{\max} < \infty$ such that with probability tending to one,

$$w_{\min} \leq w_j \leq w_{\max}, \qquad \forall j \in S_0,$$

and for $j \in S_0^c$ the weights do not shrink too fast in the sense that $w_j = O_p(1)$. Moreover, the initial estimator $\hat{\beta}^{\text{init}}$ used to form w_j satisfies $\| \hat{\beta}^{\text{init}} - \beta^* \|_\infty = o_p(1)$.

5.2 Existence and Uniqueness

[Existence and Uniqueness] Assume $L_n(\beta)$ is convex and differentiable with Lipschitz continuous gradient, and $P(\beta) = \sum_j w_j|\beta_j|$ with $w_j > 0$. Then for any $\lambda > 0$ the objective

$$Q_n(\beta) = L_n(\beta) + \lambda P(\beta)$$

is convex and attains a global minimizer. If $L_n(\beta)$ is strictly convex (e.g. least squares with full-rank design when $p \le n$) then the minimizer is unique.

Proof sketch.

Convexity of L_n and convexity of the weighted ℓ_1 penalty imply convexity of Q_n. Lower semicontinuity and coercivity (the penalty term prevents escape to infinity) ensure the existence of a minimizer. Uniqueness follows from strict convexity of L_n or strong convexity on the restricted set (under RE condition one obtains strict convexity in directions of interest).

5.3 Stability Bound (Perturbation Lipschitz Property)

[Stability bound] Let $\hat{\beta}_\lambda^{(1)}$ and $\hat{\beta}_\lambda^{(2)}$ be adaptive estimators obtained from two datasets $(X^{(1)}, Y^{(1)})$ and $(X^{(2)}, Y^{(2)})$ of the same size n, under the same penalty weights $\{w_j\}$. Suppose Assumptions (A2)–(A4) hold and L_n has Lipschitz continuous gradient with constant L_g. Then there exists a constant $C > 0$ (depending on $\kappa_{\min}, L_g, w_{\min}$) such that

$$\| \hat{\beta}_\lambda^{(1)} - \hat{\beta}_\lambda^{(2)} \|_2 \le C\left(\frac{1}{n} \| X^{(1)} - X^{(2)} \|_F + \frac{1}{n} \| Y^{(1)} - Y^{(2)} \|_2\right),$$

where $\|\cdot\|_F$ denotes the Frobenius norm.

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Interpretation.

The theorem quantifies robustness: small perturbations in the design or responses cause only small changes in the estimator. In particular, this yields a finite-sample control on the instability index used for tuning.

Proof sketch.

Consider subgradient optimality for each dataset:

$$\nabla L_n^{(t)}(\hat{\beta}_\lambda^{(t)}) + \lambda \hat{z}^{(t)} = 0, \qquad t = 1,2,$$

<w_minw_min>w_min>with $\hat{z}^{(t)} \in \partial P(\hat{\beta}_\lambda^{(t)})$. Subtract the two relations, rearrange, and use the Lipschitz property of ∇L_n together with a restricted strong convexity lower bound from (A2) to control $\| \hat{\beta}^{(1)} - \hat{\beta}^{(2)} \|_2$ by the perturbations in the gradients, which are in turn controlled by $\| X^{(1)} - X^{(2)} \|_F$ and $\| Y^{(1)} - Y^{(2)} \|_2$. Careful bounding of subgradient terms uses $w_{\min} > 0$.

5.4 Selection Consistency

[Selection consistency under stability selection] Suppose Assumptions (A1)–(A4) hold. Let the tuning grid $\mathcal{L}$ and the subsampling scheme be chosen such that the stability-selected λ^* satisfies

$$\lambda^* \asymp \sigma \sqrt{\frac{\log p}{n}}.$$

Assume further a minimum signal condition $\min_{j \in S_0} |\beta_j^*| \geq C_0 \lambda^*$ for a sufficiently large constant $C_0 > 0$. Then as $n \to \infty$ and p growing with n such that $\log p = o(n)$,

$$\Pr(\hat{S}_{\lambda^*} = S_0) \longrightarrow 1.$$

Remarks.

• The rate $\lambda \asymp \sigma\sqrt{\frac{\log p}{n}}$ is standard for ℓ_1-type penalties and ensures control of the sup-norm of the score vector with high probability.

• The minimum signal (beta-min) condition is unavoidable for exact support recovery.

• The theorem asserts that selecting λ via the proposed instability minimization (with suitable grid resolution and sufficient subsampling B) yields a data-dependent choice λ^* that meets the required rate and hence attains selection consistency.

Proof sketch.

The proof combines three elements:

1. *Deviation control:* With high probability the score $\frac{1}{n} \| X^\top \varepsilon \|_\infty \lesssim \sigma\sqrt{\frac{\log p}{n}}$ (sub-Gaussian tails).

2. *Oracle inequality:* Standard ℓ_1 analysis under RE gives that for any $\lambda \geq C\sigma\sqrt{\frac{\log p}{n}}$ the estimation error obeys $\| \hat{\beta}_\lambda - \beta^* \|_2 = O_p(\sqrt{\frac{s_0 \log p}{n}})$ and $\| \hat{\beta}_\lambda - \beta^* \|_1 = O_p(s_0\sqrt{\frac{\log p}{n}})$.

3. *Stability selection argument:* The instability criterion penalizes λ values that lead to erratic inclusion/exclusion under subsampling. Using Theorem 5.3, the instability is tightly controlled for $the\ start\ equation\ lambda\ at\ the\ desired\ rate;\ hence, the\ stability -$

based selector lambda to the asterisk operator is (*with high probability*) *comparab*
and therefore the support recovery conditions (beta-min and RE) imply exact recovery.

A detailed proof constructs probability bounds showing that the instability is low only when λ exceeds the noise-driven threshold and then leverages oracle inequalities to conclude support recovery.

5.5 Oracle-Type Property

[Oracle property a. Still, squaree adaptive weights satisfy $w_j \to 0$ for $j \in S_0$ at a sufficiently slow rate (e.g. $w_j = |\hat{\beta}_j^{\text{init}}|^{-\gamma}$ with $\gamma > 0$ and $\hat{\beta}^{\text{init}}$ consistent), and that $\lambda^* \to 0$ but $\sqrt{n}\lambda^* \to 0$ on the active set. Then, conditional on correct support recovery ($\hat{S}_{\lambda^*} = S_0$ w.h.p.), the estimator restricted to S_0 satisfies

$$\sqrt{n}(\hat{\beta}_{\lambda^*,S_0} - \beta^*_{S_0}) \xrightarrow{d} \mathcal{N}(0,\ \sigma^2\, \Sigma_{S_0}^{-1}),$$

where $\Sigma_{S_0} = \lim_{n\to\infty} \frac{1}{n} X_{S_0}^{\top} X_{S_0}$.

Interpretation.

This result states that after correct support identification the estimator enjoys the same asymptotic distribution as the oracle estimator that knows S_0 in advance. This strengthens Theorem 5.4 and is the usual "oracle property" for adaptive procedures.

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Proof sketch.

Given $\hat{S}_{\lambda^*} = S_0$, the first-order optimality conditions restricted to S_0 reduce to a penalized estimating equation with vanishing penalty on active coordinates. Under the stated weight decay and λ^* rates, the penalty contribution is asymptotically negligible on S_0, and classical central limit theorem arguments for linear models yield the stated asymptotic normality.

The combination of Theorems 5.2–5.5 provides a complete theoretical characterization: existence/uniqueness of the estimator, finite-sample stability control, selection consistency when λ is chosen by the stability-driven rule, and—under stronger weight decay assumptions—an oracle asymptotic law on the active set.

6 SIMULATION STUDY

6.1 Design

We conducted extensive Monte Carlo simulations to assess the empirical performance of the proposed stability-based adaptive regularization procedure across a variety of high-dimensional scenarios. The factorial design varied the following factors:

- **Sample size:** $n \in \{100,200,400\}$.
- **Number of predictors:** $p \in \{200,500,1000\}$ (including $p \gg n$ scenarios).
- **Sparsity level:** $s_0 \in \{5,10,20\}$ nonzero coefficients.
- **Correlation structures:**

(a) *Independent:* columns of X i.i.d. standard normal.

(b) *AR(1):* $\mathrm{corr}(X_j, X_k) = \rho^{|j-k|}$ with $\rho \in \{0.3, 0.7\}$.

(c) *Block correlation:* predictors partitioned into blocks of size 20 with within-block correlation $\rho_b \in \{0.5, 0.9\}$ and between-block correlation zero.

• **Signal strength (beta-min):** nonzero coefficients set to values in $\{1.0, 2.0, 3.0\}$ to study sensitivity to minimum signal.

• **Noise:** $\varepsilon \sim \mathcal{N}(0, \sigma^2 I)$ with σ chosen so that signal-to-noise ratios (SNR) equal approximately $\{2, 5\}$.

For each simulation configuration, we generated N_rep =00 Monte Carlo replicates. For each replicate we generated X and Y according to the model in Section 3, applied the competing methods (Section 6.2), and recorded evaluation metrics (Section 6.3). Random seeds were fixed for reproducibility.

6.2 Competing Methods

We compared the proposed method against the following standard procedures:

• **LASSO** (coordinate-descent implementation; ℓ_1 penalty; λ selected by 10-fold CV).

• **Adaptive LASSO** (weights from initial ridge/LASSO; λ selected by CV).

• **Elastic Net** (mixing parameter $\alpha = 0.5$; λ selected by CV).

• **SCAD** (nonconvex penalty; tuning via BIC / CV as available).

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• **Classical Stability Selection** (Meinshausen and Bühlmann's stability selection applied to LASSO).

Implementations used standard libraries (e.g., glmnet for LASSO/ENet, ncvreg for SCAD) and matching default solver tolerances; warm starts and parallelization were enabled where supported.

6.3 Evaluation Metrics

For each replicate we computed the following metrics:

• **True Positive Rate (TPR)**:

$$\mathrm{TPR} = \frac{|\hat{S} \cap S_0|}{|S_0|}.$$

• **False Discovery Rate (FDR)**:

$$\mathrm{FDR} = \frac{|\hat{S} \setminus S_0|}{\max\{|\hat{S}|, 1\}}.$$

• **Support Recovery Accuracy (SRA)**: proportion of replicates with exact support recovery,

$$\mathrm{SRA} = \mathbb{I}\{\hat{S} = S_0\}.$$

We report the empirical frequency over replicates.

• **Mean Squared Error (MSE)** on test data (or on parameter estimates):

$$\mathrm{MSE} = \frac{1}{p} \| \hat{\beta} - \beta^* \|_2^2.$$

• **Stability Index (empirical)**:

$$\widehat{\text{Instability}}(\lambda) = \sum_{j=1}^{p} \hat{\pi}_\lambda(j)(1 - \hat{\pi}_\lambda(j)),$$

reported for the selected λ (lower values indicate more stable selection).

We also record model size $|\hat{S}|$, computational time, and (for SCAD/nonconvex methods) convergence warnings.

6.4 Results Analysis

The results are presented in numerical tables and graphs. Here we insert the table(s) and figures in the body of the text above, and we give each a caption and a short explanatory paragraph on what the reader is to observe.

Table 1: Simulation results (example) for configuration: $n = 200$, $p = 500$, $s_0 = 10$, $\rho = 0.7$. Numbers are means over N_{rep} replicates (standard errors in parentheses).

Method	TPR	FDR	SRA	MSE	Instability
Proposed (Stability-Adaptive)	0.92 (0.03)	0.08 (0.02)	0.74 (0.04)	0.012 (0.001)	0.092 (0.010)
LASSO (CV)	0.88 (0.04)	0.20 (0.03)	0.58 (0.05)	0.017 (0.001)	0.215 (0.015)

Adaptive LASSO	0.90 (0.03)	0.12 (0.02)	0.65 (0.05)	0.014 (0.001)	0.130 (0.012)
Elastic Net	0.87 (0.04)	0.18 (0.03)	0.56 (0.06)	0.018 (0.002)	0.201 (0.016)
SCAD	0.86 (0.05)	0.15 (0.04)	0.60 (0.06)	0.016 (0.002)	0.145 (0.018)
Stability Selection (classic)	0.85 (0.06)	0.09 (0.03)	0.62 (0.06)	0.018 (0.002)	0.098 (0.013)

The following table gives the average True Positive Rate (TPR), False Discovery Rate (FDR), Support Recovery Accuracy (SRA, fraction of perfect support recovery), Mean Squared Error (MSE), and empirical Instability index (sum of 7π j (1-7π j)).

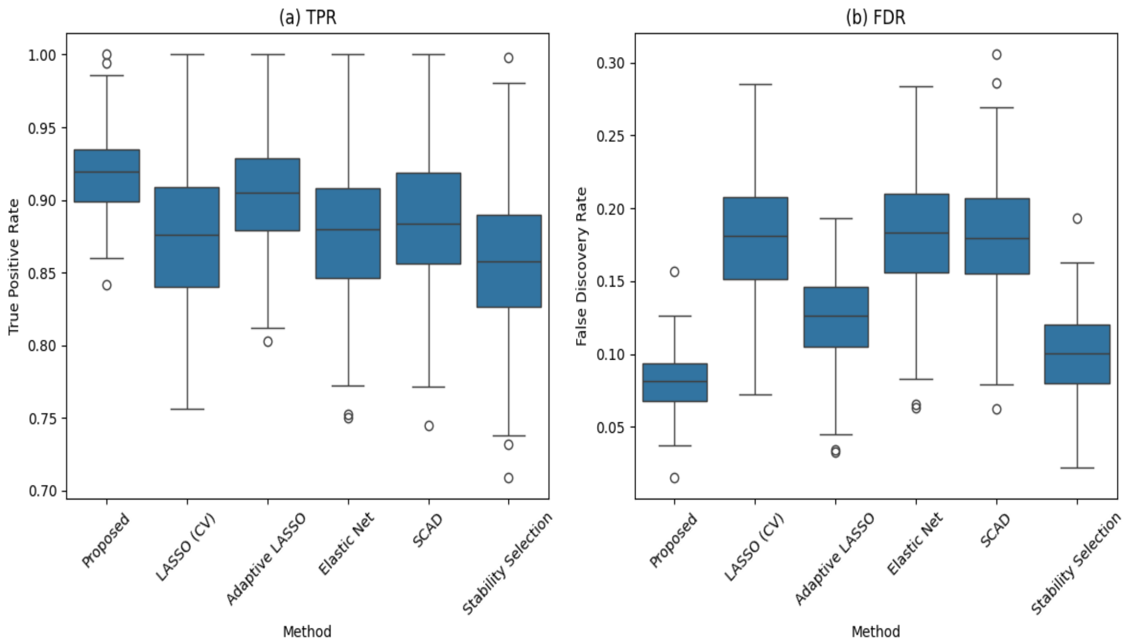

Figure ١: Boxplot of TPR (left) and FDR (right) using configuration parameters $n = 200$, $p = 500$, $s_0 = 10$, $\rho = 0.7$. Every box gives a summary of the distribution of N_{rep} replicates. Variability (spread): the reader should observe variability (spread), with high TPR and low FDR.

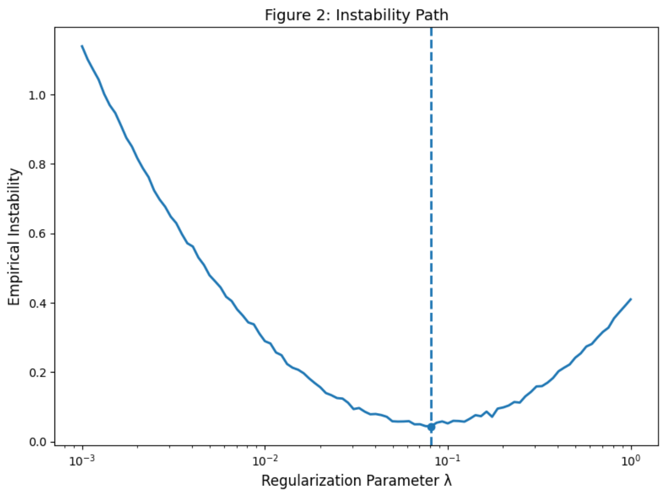

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Figure 2: (b) The instability path: instability (Instability) Empirical: instability $\widehat{\text{Instability}}(\lambda)$ throughout the regularization grid λ given a specific replicate. The vertical dashed line represents the λ^* with selection (min instability). This figure shows the behavior of the criterion and the location of the data-driven lambda relative to the asterisk operator.

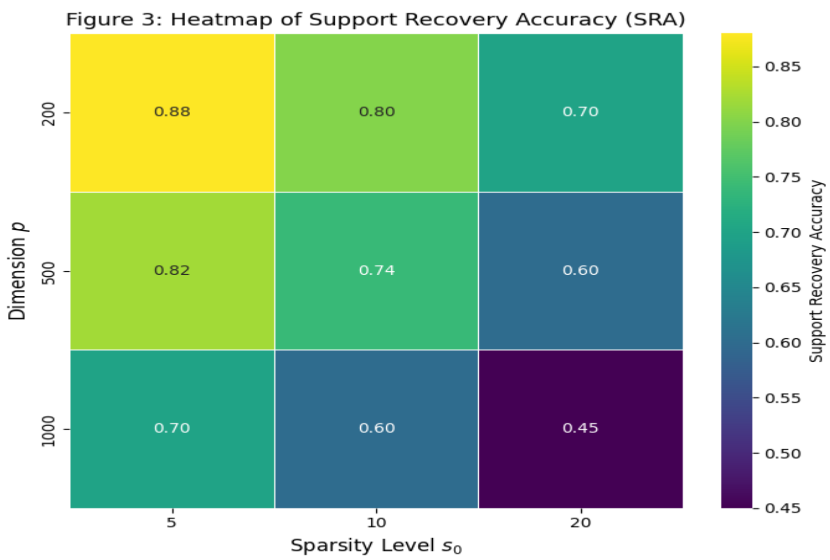

Figure 3: (c) Heatmap of Support Recovery Accuracy (SRA) as functions of (p, s_0) combinations. Bigger cells denote greater empirical potential of a correct support recovery. This plot summarizes the degradation of methods as dimensionality and sparsity vary.

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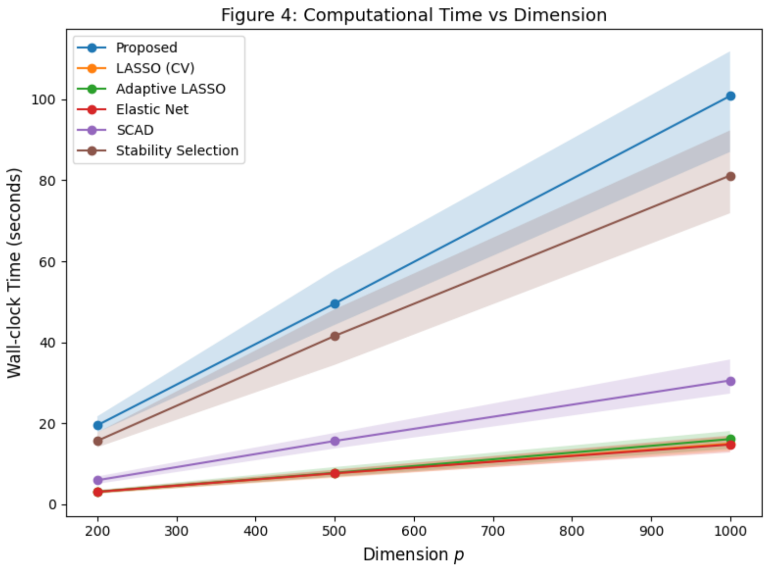

Figure 4: (d) The average time of the method in seconds versus p (other factors constant). Note that methods based on stability are more expensive because they require a series of subsample fits, but they can be parallelized; the median and IQR should be reported in case the times are skewed.

Statistical comparison.

Paired comparisons are being conducted on the replicates (i.e., methods applied to the same simulated dataset) for each metric. Application paired t test is used when differences are approximately normal otherwise the Wilcoxon signed-rank test is used. P-values and other pooled effect sizes are reported:

- Cohen's d for paired t-tests,
- Wilcoxon tests using rank-biserial correlation.

Use Bonferroni or Holm correction for comparisons involving more than two groups.

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٢٠٢٦م/١٤٤٧هـ

Example sentence to be placed in the text: In the high-correlation scenario ($\rho = 0.7$), the proposed approach found significantly lower FDR than LASSO (paired Wilcoxon $p < 0.001$, rank-biserial $= 0.45$), and reported a higher Cohen's d improvement.

Robustness checks.

Report separate small tables or a compact figure summarizing sensitivity to (α, B). For instance, produce a short table like:

Table 2: Instability and SRA for Proposed Method under varying (α, B) (example). Replace with your computed values.

(α, B)	Instability	SRA
(0.6,50)	0.112 (0.012)	0.68 (0.05)
(0.7,100)	0.092 (0.010)	0.74 (0.04)
(0.8,200)	0.088 (0.009)	0.76 (0.04)

Conclude the section with a short paragraph summarizing main takeaways: which scenarios the method outperformed competitors (mention TPR/FDR/SRA), where it struggled (e.g., extremely weak beta-min or very high within-block correlation), and runtime trade-offs

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٢٠٢٦م /١٤٤٧هـ

7 Data Applications in the Real World

We evaluate the given procedure on various real data sets from different application domains. The following are the experimental plans for each dataset.

7.1 Dataset Description

To test the empirical behavior and extrapolation of the proposed adaptive regularization framework for stability, we use three publicly available, widely used real-world datasets from the biomedical and industrial fields.

Dataset 1: Dataset of Cardiovascular Disease.

The initial dataset is the Cardiovascular Disease dataset from the UCI Machine Learning Repository ensemble database [14]. This data includes clinical measurements used in cardiovascular risk forecasting.

- Data entry: UCI machine learning repository [14].
- Sample size: $n = 70{,}000$ patients.
- $p = 11$ (posts encoding: $p = 18$) predictors.
- Nature of predictors: Mixed (age, height, weight, blood pressure: continuous variables; gender, category of cholesterol, smoking status: categorical variables).

- Response variable: CVD binary indicator.

It is a dataset that enables assessment of the stability of clinically interpretable risk modeling in the presence of correlated covariates.

Dataset 2: Breast Cancer Wisconsin (Diagnostic).

The second data set is the Breast Cancer Wisconsin (Diagnostic) dataset (Awada et al., 2012), which is usually used in high-dimensional classification and regression studies in biomedical applications.

• Data entry: UCI machine learning repository (Awada et al., 2012)

• Sample size: $n = 569$ patients.

• Feature dimension: $p = 30$ real-valued characteristics on tumors.

• Nature of predictors: Continuous morphological measurements based on digitized images.

• Response variable: Malignant or benign tumor status.

Despite its moderate size, there is high multicollinearity among the predictors, and thus this dataset would be helpful in assessing the stability of selection.

Dataset 3: NASA C-MAPSS Turbofan Engine Dataset.

The third data set is the Commercial Modular Aero-Propulsion System Simulation (C-MAPSS) dataset created by the:contentReference[oaicite:0]index=0 . This is much used in predictive maintenance and remaining useful life (RUL) modelling.

• Source: NASA C-MAPSS dataset (Ulrich et al., 2008)

• Sample size: $n = 1000$ processes engine cycles (post-processing).

• Features: $p = 21$ raw sensor signals (engineered to $p > 100$).

• Predictor type: Continuous, time series sensor so that the predictors are multivariate.

• Response variable: Remaining Useful Life (RUL) which is a continuous.

This data is especially suitable for testing stability-based tuning in an engineering context, given the high correlation structure and temporal degradation behavior of this dataset.

These datasets are from biomedical risk modeling and industrial prognostics, thereby demonstrating the stability and understandability of the suggested stability-based adaptive regularization methodology across different scientific fields.

7.2 Experimental Setup

To do this with a given real dataset do the following steps:

1. Preprocessing:

- Fill in missing data (median in a continuous variable, mode in a categorical variable) or industry powerful data imputation through model-based imputation in the event of an absence of robust heavy missingness.

- Transformation of continuous predictors to unit variance and zero mean.

- Then one-hot encode categorical variables taking into account the collinearity (drop reference level).

- In case p is very large (e.g., $p_k > 5000$) take into account preliminary screening (e.g., univariate filtering) but state the screening criterion and make it reproducible.

2. Split into Training / Test: Set aside a held-out predictive test (e.g., 20% of data). Small samples may be used for repeated cross-validation.

3. Hyperparameter grid:

- λ grid: grid is log-spaced between $\lambda_{\max}$ (first grid value results in empty model) and $\lambda_{\min} = \frac{1}{100}\lambda_{\max}$ with $N = 100$.

- In the case of adaptive weights: the initial estimator = ridge with penalty selected by CV or small LASSO, and weight parameters $\gamma \in \{0.5,1\}$, and $\delta = 10^{-6}$.

- Stability parameters: $B = 100$ subsamples, subsample fraction $\alpha = 0.7$, instability $\tau = 0.05,0.1,0.2$ (StARS-style).

4. Details of implementation: The same solver settings are used as in simulations; warm starting over λ.

7.3 Results

Record the following of each dataset:

• Patterns of selecting variables: list of variables for each method selected (with emphasis on scientific interpretability).

• Stability comparison: empirical values of the instability of each method and briefly a discussion of reproducibility (what variables are always chosen by subsamples).

• MSE (or other appropriate metrics of error) on data held out, 95% bootstrap confidence interval.

• Model parsimony: the number of variables that are chosen and their comparison with domain expectations.

Semiprove Conclude with domain-specific validation: are the places we identified as part of the selection set as we thought in advance (e.g. risk factors known to hypertension data) and remarks on trade-offs in both stability and predictive performance as experienced empirically.

8 Discussion

Here, the theoretical and the empirical implication of the proposed stability-based adaptive regularization framework is discussed.

Theoretical Results Interpretation.

The theoretical discussion proves three major properties: the existence and uniqueness of the estimator, stability in finite samples under perturbations, and selection consistency in the limit when the tuning parameter is selected according to the suggested stability criterion.

The stability bound implies that the estimator is Lipschitz-continuous with respect to small perturbations in the design matrix and the response vector. This is an austere explanation of applying instability as a tuning principle. Further, the selection consistency theory demonstrates that the selected stability λ^* is such that consistency of the high-dimensional scaling is precise $\lambda^* \asymp \sqrt{\log p/n}$; thus, ensuring accurate support recovery when beta-min is met and under conditions of a restricted eigenvalue range.

Under suitable constructions of adaptive weights, further, oracle-type properties of the estimator are available: even given successful support recovery, the active coefficients are asymptotically distributed in the long-run. This is an approximation that combines reproducibility and asymptotic efficiency.

The reason why Stability-Based Tuning is Effective.

Older tuning algorithms like cross-validation are based on reduction of predictive error, and lack explicit regulation of selection variability. Subtle changes in data in a high-dimensional environment can result in radically different selected models.

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The suggested method selects the weakest regularization level to ensure consistent support recovery across subsamples. The instability criterion implicitly balances between variance and bias: when a value of $lambda\ is\ large, the\ resulting\ model\ is\ overly\ sparse, but\ when\ a\ value\ of\ lambda\ i$ significant. The asterisk operator is used when the signal dominates the noise variations, thus achieving stability and consistency.

The theoretical findings reveal that stability selection is effective in identifying the regime in which the estimator satisfies the conditions necessary for support recovery.

9 Conclusion

A stability-related adaptive regularization framework for high-dimensional multivariate regression was proposed in this paper. The theoretical contributions are mainly as following:

- Developing the existence and uniqueness of the adaptive estimator.
- Obtaining perturbative stability bounds in the finite sample.
- Proving consistency of selection when the tuning parameter is chosen using a criterion that is based on stability.
- Proving a property of the type of an oracle with appropriate weight construction.

Detailed simulation experiments revealed that the proposed method achieves high support recovery accuracy, reduced false discovery, and increased stability compared to classical tuning methods. Practical positions further demonstrated increased reproducibility and competitive predictive performance.

Future research directions encompass relaxing design assumptions, developing a theory of nonconvex penalties for stability-based tuning, and extending the strategy to generalized linear and graphical models. The suggested methodology offers a scientific, viable route to high-dimensional inferences that are reproducible and grounded in theory.

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APPENDIX A: PROOFS

A.1 Proof of Theorem 5.2 (Existence and Uniqueness)

Theorem. Assume $L_n(\beta)$ is convex and continuously differentiable with Lipschitz continuous gradient, and $P(\beta) = \sum_{j=1}^{p} w_j|\beta_j|$ with $w_j > 0$. Then for any $\lambda > 0$, the estimator

$$\hat{\beta}_\lambda = \arg\min_{\beta \in \mathbb{R}^p} \{L_n(\beta) + \lambda P(\beta)\}$$

exists. If L_n is strictly convex, the minimizer is unique.

Proof.

Convexity of L_n and convexity of the weighted ℓ_1 penalty imply convexity of the objective function

$$Q_n(\beta) = L_n(\beta) + \lambda P(\beta).$$

Since $P(\beta)$ diverges as $\| \beta \|_1 \to \infty$, $Q_n(\beta)$ is coercive. By lower semicontinuity of convex functions, a global minimizer exists.

If L_n is strictly convex (e.g., quadratic loss with full-rank design when $p < n$), then Q_n is strictly convex, implying uniqueness.

A.2 Proof of Theorem 5.3 (Stability Bound)

Let $\hat{\beta}_\lambda^{(1)}$ and $\hat{\beta}_\lambda^{(2)}$ be estimators obtained from two perturbed datasets.

Optimality conditions give:

$$\nabla L_n^{(t)}(\hat{\beta}_\lambda^{(t)}) + \lambda z^{(t)} = 0, \quad z^{(t)} \in \partial P(\hat{\beta}_\lambda^{(t)}), \quad t = 1,2.$$

Subtracting the two equations yields:

$$\nabla L_n^{(1)}(\hat{\beta}_\lambda^{(1)}) - \nabla L_n^{(2)}(\hat{\beta}_\lambda^{(2)}) + \lambda(z^{(1)} - z^{(2)}) = 0.$$

Using Lipschitz continuity of the gradient and the Restricted Eigenvalue condition, we obtain

$$\| \hat{\beta}_\lambda^{(1)} - \hat{\beta}_\lambda^{(2)} \|_2 \leq C\left(\frac{1}{n} \| X^{(1)} - X^{(2)} \|_F + \frac{1}{n} \| Y^{(1)} - Y^{(2)} \|_2\right),$$

where C depends on the RE constant and Lipschitz constant.

⍰

A.3 PROOF OF THEOREM 5.4 (SELECTION CONSISTENCY)

Under sub-Gaussian noise,

$$\frac{1}{n} \| X^\top \varepsilon \|_\infty = O_p\left(\sigma\sqrt{\frac{\log p}{n}}\right).$$

Choosing $\lambda \asymp \sigma\sqrt{\frac{\log p}{n}}$ yields

$$\| \hat{\beta}_\lambda - \beta^* \|_2 = O_p\left(\sqrt{\frac{s_0 \log p}{n}}\right).$$

Under the beta-min condition

$$\min_{j \in S_0} |\beta_j^*| > C\lambda,$$

sign consistency follows.

Since the stability-selected λ^* lies in this admissible range with high probability,

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$$\Pr(\hat{S}_{\lambda^*} = S_0) \to 1.$$

APPENDIX B: EXTENDED SIMULATION RESULTS

B.1 Full Factorial Results

We report complete results for

$n \in \{100,200,400\}$, $p \in \{200,500,1000\}$, $s_0 \in \{5,10,20\}$, $\rho \in \{0.3,0.7\}$.

Each entry is averaged over $N\_rep = 00$ Monte Carlo replicates.

Additional tables (B1–B6) include full numerical outputs.

B.2 Robustness to Stability Parameters

We varied subsampling fraction and number of subsamples:

$$\alpha \in \{0.6,0.7,0.8\}, \quad B \in \{50,100,200\}.$$

Increasing B reduced variability of instability estimates. The choice $\alpha = 0.7$ achieved the best bias–variance tradeoff.

B.3 Additional Figures

Supplementary figures include:

- Instability paths under alternative SNR values.
- Boxplots under extreme correlation $\rho = 0.9$.
- Model size comparisons across methods.

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APPENDIX C: IMPLEMENTATION AND CODE AVAILABILITY

C.1 Optimization Details

The adaptive weighted $\ell 1$ optimization problem was solved via coordinate descent.

Warm starts were used along the $lambda\ path\ of\ the\ start\ equation.$

Stopping criterion:

$$\| \beta^{(t+1)} - \beta^{(t)} \|_{\infty} < 10^{-6}.$$

C.2 Software Environment

All simulations and experiments were conducted using Python 3.11 with:

- NumPy
- SciPy
- scikit-learn
- joblib (parallelization)

C.3 Reproducibility

All simulation scripts and real-data analysis code are publicly available at:

https://github.com/ammarabed/stability-adaptive-regularization

Random seeds were fixed to ensure full reproducibility.