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العدد الثالث
والأربعون

تصنيف فضاءات مارغوليس في فضاء مينكوفسكي ثلاثي الأبعاد

إبراهيم محمد خلف

رياضيات تبلوجي\ جامعة مراغة\ كلية العلوم\ قسم الرياضيات\ ايران

ibraheemmohammed303@gmail.com

المستخلص:

تقدّم هذه النسخة المنقحة إطارًا تصنيفيًا مضبوطًا لفضاءات مارغوليس ذات الجزء الخطي الثابت $L: \Gamma \to SO_0$(٢،١) عندما يكون L محدبًا–متراصًا (وبالتالي أنوسوف في الرتبة الأولى). التحسين الجوهري في هذه المراجعة هو أن مخاريط القبول لا تُعرَّف بعد الآن على أنها تقاطع غير مُفسَّر لعدد لا نهائي من أنصاف الفراغات، بل تُعرَّف بواسطة الدالية المستمرة التي أنشأها Margulis–Labourie–Goldman على فضاء التيارات الجيوديسية؛ ومن ثم نحصل مباشرة على أن المخروطين C_L^+ و C_L^- مفتوحان ومحدبان. كما أُزيل افتراض "ثبات الإشارة" من صياغة النظرية الرئيسة، وأصبح نتيجةً مستخرجة من لا–انعدام الدالية على فضاء التيارات واتصال هذا الفضاء. وتستبدل البراهينُ المنقحةُ القفزةَ السابقة من إيجابية ثابت مارغوليس إلى "السماحية" الهندسية بمرجعين محددين: معيار Margulis–Labourie–Goldmann للإجراء الصحيح، ونظرية Kassel–Guéritaud–Danciger التي تعطي مجالات أساسية محددة بمستويات معوجة. وفي القسم التطبيقي أُضيف مثال صريح لزمرة شوتكي من الرتبة ٢: تُكتب المصفوفات $(L(a), L(b))$ صراحةً، ويُحسب البعد $\dim H^1$ = ٣، وتُستخرج متباينات مارغوليس للعناصر a و b و ab في إحداثيات صريحة، مع رسم للمخروط المقطوع الناتج في R^3. وبذلك تعالج النسخة الحالية الملاحظات البنيوية والإثباتية والأسلوبية التي أثارها التحكيم، مع الإبقاء على الهدف المركزي للبحث ضمن سياق مضبوط وقابل للتحقق.

الكلمات المفتاحية: فضاءات مارغوليس؛ فضاء مينكوفسكي؛ الأفعال الأفينية الصحيحة؛ ثابت مارغوليس؛ علم التماثل الجماعي؛ التيارات الجيوديسية؛ المستويات المعوجة.

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Classification of Margulis Spacetimes in Three-Dimensional Minkowski Space

Ibrahim Mohammed Khalaf

Tabluji Mathematics / Maragheh University / College of Science/ Department of Mathematics, Iran

ibraheemmohammed303@gmail.com

Abstract:

This revised manuscript gives a precise classification framework for Margulis spacetimes with fixed convex-cocompact linear holonomy L: $\Gamma \rightarrow SO_0(2,1)$. The key structural point is that the space of affine deformations is $H^1(\Gamma, R^{2,1}_L)$ and that properness is detected by the Goldman-Labourie-Margulis functional Ψ_L on geodesic currents. This makes the admissible set a pair of open convex cones C_L^+ and C_L^- in cohomology, and it removes the unsupported "sign-stability" hypothesis from the earlier version: constant sign becomes a theorem rather than an assumption. The revised proofs no longer infer crooked-plane allowability directly from positivity of generator-wise Margulis invariants; instead, they combine the current-theoretic properness criterion of Goldman-Labourie-Margulis with the crooked-plane realization theorem of Danciger-Guéritaud-Kassel. We also formulate a fixed-linear-part marked Margulis spectrum rigidity statement and provide a fully explicit rank-two Schottky example. In that example, the matrices L(a) and L(b) are written down, dim $H^1 = 3$ is computed, the Margulis inequalities for a, b, and ab are made explicit, and a truncated cone in R^3 is drawn. The manuscript is now written in a single main language (English), carries numbered references, and isolates the exact places where established results from the literature enter the proofs.

Keywords: Margulis spacetime; Minkowski space; proper affine action; Margulis invariant; group cohomology; geodesic current; crooked plane.

1. Introduction

A Margulis spacetime is a quotient $M = \Gamma_{L,u} \setminus E$ of three-dimensional Minkowski space $E = R^{2,1}$ by a free, properly discontinuous affine action of a nonabelian free group. Such manifolds were the first known nonamenable examples in affine geometry and form a central chapter in the story surrounding the Auslander conjecture. Their geometry sits at the meeting point of Lorentzian geometry, hyperbolic dynamics, affine deformation theory, and geometric group theory. The aim of the present revision is not to enlarge the scope of the original manuscript, but to place its central classification statement on a mathematically precise and fully referenced foundation.

The referee correctly identified four structural issues in the earlier version: (i) the admissible cone was described informally as the intersection of infinitely many linear inequalities without explaining why this defines an open set in finite-dimensional cohomology; (ii) the constant-sign property of Margulis invariants was used as an assumption rather than derived from a precise theorem; (iii) the proofs in Section 5 were sketches, especially at the two delicate points where one passes from positivity of Margulis invariants to properness and from properness to the impossibility of sign oscillation; and (iv) the example section did not contain an explicit rank-two computation. The present version addresses each point directly.

Two changes are conceptually decisive. First, the admissible cones are defined through the continuous functional on geodesic currents introduced by Goldman, Labourie, and Margulis [7], rather than through a naive countable intersection. This immediately explains openness, convexity, and sign coherence. Second, the existence of finite crooked-plane fundamental domains is not deduced from an unsupported generator-wise allowability implication; instead, it is imported from the precise crooked-plane theorem of Danciger, Guéritaud, and Kassel [2, Theorems 1.7 and 1.8]. The resulting manuscript is mathematically more modest in its claims, but substantially stronger in its proofs.

1.1. MAIN STATEMENTS IN REVISED FORM

1. For a fixed faithful convex-cocompact representation $L: \Gamma \to SO_0(2,1)$, the affine deformation space with fixed linear part is $H^1(\Gamma, R^{2,1}_L)$, and proper affine actions are exactly the cohomology classes in the two admissible cones C_L^+ and C_L^-.
2. Up to marked affine equivalence, the moduli space is $C_L^+ \cup C_L^-$; up to unmarked affine conjugacy, it is the quotient of this set by the centralizer $Z_{SO_0(2,1)}(L)$.
3. For the fixed linear part, equality of the marked Margulis spectrum forces equality of cohomology classes; hence, the affine actions are translation-conjugate.
4. A fully explicit Schottky example of rank two is worked out in coordinates, including matrices, cohomology dimension, three basic Margulis inequalities, and a picture of the resulting truncated cone.

1.2. SCOPE AND ORGANIZATION

Section 2 fixes notation and clarifies the three notions that were previously left ambiguous: convex-cocompactness in $SO_0(2,1)$, the canonical normalization of the neutral vector v_γ^0, and the definition of the cone C_L via geodesic currents. Section 3 records the cohomological parametrization and the quotient by the centralizer. Section 4 states the revised main theorems. Section 5 contains complete proofs assembled from the current criterion of [7] and the crooked-plane results of [2]. Section 6 provides the explicit rank-two example requested by the referee. A numbered reference list closes the paper.

2. Preliminaries and corrected definitions

2.1. MINKOWSKI SPACE AND AFFINE DEFORMATIONS

Let $E = R^{2,1}$ be the affine space modeled on R^3 equipped with the Lorentzian bilinear form $B(x,y,z) = x^2 + y^2 - z^2$. The identity component of its linear isometry group is $SO_0(2,1)$, and the orientation− and time−

orientation−preserving affine isometry group is $SO_0(2,1) \ltimes R^{2,1}$. Fix a nonabelian free group Γ of finite rank $r \geq 2$ and a representation

$$\rho_{u(\gamma)})(x) = L(\gamma)x + u(\gamma), \qquad u(\gamma\delta) = u(\gamma) + L(\gamma)u(\delta).$$

Here $L\colon \Gamma \to SO_0(2,1)$ is the linear holonomy and $u\colon \Gamma \to R^{2,1}$ is an L−cocycle. Two cocycles differing by a coboundary determine affine actions conjugate by a translation. Therefore, the translation−conjugacy class of the affine deformation depends only on $[u] \in H^1(\Gamma, R^{2,1}_L)$.

2.2. CONVEX−COCOMPACT VERSUS SCHOTTKY

Throughout the paper, the linear holonomy L is assumed to be faithful, discrete, and convex−cocompact. Concretely, this means that $L(\Gamma)$ contains no parabolic element and that the hyperbolic surface $\Sigma_L = L(\Gamma) \setminus H^2$ has compact convex core. In the rank−one group $SO_0(2,1)$, convex−cocompactness is equivalent to the Anosov property. When Γ is free, one may choose generators so that $L(\Gamma)$ is presented as a classical Schottky group; conversely, every classical Schottky subgroup is convex−cocompact. This is the precise sense in which the words “convex−cocompact”, “Anosov”, and “Schottky” are linked in the present setting [4, 7].

2.3. NEUTRAL VECTORS AND THE MARGULIS INVARIANT

For a hyperbolic element $g = L(\gamma) \in SO_0(2,1)$, let v_g^+ and v_g^- be future−pointing lightlike eigenvectors associated to the eigenvalues $\lambda(g) > 1$ and $\lambda(g)^{-1}$, normalized by $B(v_g^+, v_g^-) = -1$. There is then a unique spacelike unit vector v_g^0 satisfying $B(v_g^0, v_g^0) = 1$ and such that the ordered triple (v_g^+, v_g^0, v_g^-) has positive orientation. This removes the ambiguity noted by the referee: the “neutral vector” is not an arbitrary unit vector on a line, but a canonically normalized one depending continuously on g. The Margulis invariant of a cocycle u is

$$\alpha_u(\gamma) = B\big(u(\gamma), v^0_{L(\gamma)}\big).$$

Because every nontrivial element of a convex–cocompact free subgroup of $SO_0(2,1)$ is hyperbolic, $\alpha_{u(\gamma)}$ is defined for every $\gamma \neq e$. Standard properties recalled in Goldman's survey [6] imply that $\alpha_{u(\gamma^n)} = |n|\ \alpha_{u(\gamma)}$ and that $\alpha_{u(\gamma)} = 0$ exactly when the affine isometry $\rho_{u(\gamma)}$ has a fixed point.

2.4. THE ADMISSIBLE CONES C_L^+ AND C_L^-

The referee correctly objected that the earlier definition of C_L as an infinite intersection of open half–spaces did not explain why the resulting set is open in finite–dimensional cohomology. The clean way to formulate the cone is to use geodesic currents. Let $C(\Sigma_L)$ be the compact connected space of unit geodesic currents on Σ_L. Goldman, Labourie, and Margulis construct a continuous bilinear map.

$$\Psi_L : C(\Sigma_L) \times H^1\big(\Gamma, \mathbb{R}^{2,1}_L\big) \to \mathbb{R}$$

whose value on the periodic current μ_γ associated to a closed geodesic satisfies

$$\Psi_L\big(\mu_\gamma, [u]\big) = \frac{\alpha_u(\gamma)}{\ell_L(\gamma)}.$$

where $\ell_{L(\gamma)}$ is the hyperbolic length of the closed geodesic corresponding to γ [7, Introduction and Section 6]. We define

$$C_L^+ = \{\, [u] : \Psi_{L(\mu}, [u]) > 0 \text{ for all } \mu \in C(\Sigma_L) \,\},$$

$$C_L^- = \{\, [u] : \Psi_{L(\mu}, [u]) < 0 \text{ for all } \mu \in C(\Sigma_L) \,\} = -\, C_L^+.$$

Proposition 2.4 (openness and convexity of the admissible cones). The sets C_L^+ and C_L^- are open convex cones in $H^1(\Gamma, R^{2,1}_L)$. Moreover, because periodic currents are dense in $C(\Sigma_L)$, positivity on all currents is equivalent to positivity on all primitive conjugacy classes.

Proof. Let $[u_0] \in C_L^+$. Since $C(\Sigma_L)$ is compact and $\mu \mapsto \Psi_{L(\mu}, [u_0])$ is continuous, the minimum $m = \min_{\mu \in C(\Sigma\_L)} \Psi_{L(\mu}, [u_0])$ is positive. Continuity of Ψ_L in the cohomology variable then gives a neighborhood U of $[u_0]$ such that $\Psi_{L(\mu}, [u]) > m/2$ for all μ and all $[u] \in U$. Hence C_L^+ is open; the same argument applies to C_L^-. Convexity follows from linearity in [u]. Finally, periodic currents are dense in $C(\Sigma_L)$, and Ψ_L is continuous, so the sign condition on all currents is equivalent to the sign condition on all primitive conjugacy classes. This resolves the topological point raised by the referee.

3. Cohomological parametrization and conjugacy

3.1. AFFINE DEFORMATIONS AS H^1

Because Γ is free, a cocycle is determined by its values on a free generating set. Hence, $Z^1(\Gamma, R^{2,1}_L)$ is finite-dimensional, and $H^1(\Gamma, R^{2,1}_L)$ is the natural parameter space for affine deformations with fixed linear part. If $\tau_{v(x})= x + v$ is a translation, then

$$\tau_v\, \rho_{u(\gamma})\, \tau_v^{-1}(x) = L(\gamma)x + (u(\gamma) + L(\gamma)v - v),$$

So translation conjugacy changes u by the coboundary $\delta v(\gamma) = L(\gamma)v - v$. Thus, H^1 is exactly the marked affine deformation space.

3.2. THE REMAINING QUOTIENT BY THE CENTRALIZER

After quotienting by translations, one still has the possibility of conjugating by a linear isometry A that commutes with $L(\Gamma)$. Such an A acts on cocycles by $(Au)(\gamma) = A\,u(\gamma)$, hence linearly on H^1. Denote by $Z_{SO_0(2,1)}(L)$ the centralizer of $L(\Gamma)$ in $SO_0(2,1)$.

Proposition 3.2 (classification of the fixed-linear-part quotient). Two affine actions with the same linear part L are affinely conjugate if and only if their cohomology classes lie in the same $Z_{SO_0(2,1)}(L)$-orbit. Consequently, the unmarked moduli space with fixed linear holonomy is obtained from the marked one by dividing by this centralizer action.

This proposition is elementary but important: it explains precisely why “one point of the cone” corresponds to “one marked affine deformation,” and why an additional quotient is required only when one forgets the marking.

3.3. FINITE TRUNCATIONS OF THE CONE

The full admissible cone is defined by all currents, equivalently by all primitive conjugacy classes. In explicit computations, one often works with a finite truncation K_F determined by a finite set F of words. Such a truncation is useful for visualization but must not be mistaken for the full cone unless additional arguments are supplied. Section 6 follows this principled approach: the inequalities for a, b, and ab are made completely explicit, and the corresponding truncated cone is drawn inside R^3.

4. Main theorems

Theorem 4.1 (opposite-sign principle in current form). If the affine action $\Gamma_{L,u}$ acts properly discontinuously on E, then $\Psi_{L}(\mu, [u])$ has constant sign on all of $C(\Sigma_L)$. Equivalently, $\alpha_{u(\gamma)}$ has the same sign for every nontrivial $\gamma \in \Gamma$. Thus, sign stability is a theorem, not an extra assumption.

Theorem 4.2 (classification with fixed convex-cocompact linear part). Let Γ be a nonabelian free group and let $L: \Gamma \rightarrow SO_0(2,1)$ be faithful and convex-cocompact. Then for any cocycle representative u of a class $[u] \in H^1(\Gamma, R^{2,1}_L)$, the affine action $\Gamma_{L,u}$ on E is free and properly discontinuous if and only if $[u] \in C_L^+ \cup C_L^-$. Hence, the marked moduli space of Margulis spacetimes with linear holonomy L is exactly $C_L^+ \cup C_L^-$, and the unmarked moduli space is $(C_L^+ \cup C_L^-)/Z_{SO_0(2,1)}(L)$. If L is presented by a classical Schottky system, every class in $C_L^+ \cup C_L^-$ also admits a finite crooked-plane fundamental domain [2, Theorem 1.7].

Theorem 4.3 (marked-Margulis-spectrum rigidity for fixed linear part). Let u_1 and u_2 be L-cocycles. If $\alpha_{u_1}(\gamma) = \alpha_{u_2}(\gamma)$ for every nontrivial $\gamma \in \Gamma$, then $[u_1] = [u_2]$ in $H^1(\Gamma, R^{2,1}_L)$; therefore the affine actions Γ_{L,u_1} and Γ_{L,u_2} are conjugate by a translation. Because the hyperbolic lengths $\ell_{L(\gamma)}$ and the entropy h_L are fixed once L is fixed, the same conclusion holds if the normalized spectra $\alpha_{u\_i}(\gamma)/\ell_{L(\gamma)}$), or any common positive scalar multiple thereof, agree for all γ.

Proposition 4.4 (boundary of the admissible cone). The topological boundary ∂C_L^+ is contained in the union of the kernels ker $\Psi_{L(\mu}, \cdot)$ over $\mu \in C(\Sigma_L)$. In particular, any boundary class can be approximated by classes for which α_n tends to zero along a sequence of primitive words γ_n. The same description holds for C_L^-.

5. Detailed proofs of the main results

5.1. PROOF OF THEOREM 4.2

We divide the proof into four steps. The two delicate points singled out by the referee are treated explicitly: the sufficiency step no longer passes directly from $\alpha_i) > 0$ to a generator-wise allowability statement, and the necessity step no longer relies on an undeveloped alternating-word argument.

Step 1. Fix $[u] \in H^1(\Gamma, R^{2,1}_L)$ and choose a cocycle representative u. Replacing u by a cohomologous cocycle changes the affine action by translation conjugacy, so properness and freeness depend only on [u]. For every $\gamma \neq e$, the Margulis invariant $\alpha_{u(\gamma)})$ is defined because $L(\gamma)$ is hyperbolic.

Step 2 (sufficiency). Assume $[u] \in C_L^+$. Then $\Psi_{L(\mu}, [u]) > 0$ for all $\mu \in C(\Sigma_L)$. By the Goldman-Labourie-Margulis properness criterion [7, Introduction; Section 6], the affine action $\Gamma_{L,u}$ is properly discontinuous. It is also free: if some nontrivial γ fixed a point, then $\alpha_{u(\gamma)}) = 0$ by the standard Margulis-invariant property [6], whereas $\Psi_{\gamma}, [u]) > 0$ implies $\alpha_{u(\gamma)}) = \ell_{L(\gamma)}) \Psi_{\gamma}, [u]) > 0$. Therefore, $\Gamma_{L,u}$ acts freely and properly on E. The same reasoning applies to classes in C_L^- after replacing u by -u. Finally, once properness is known, the existence of a finite crooked-plane fundamental domain follows from [2, Theorem 1.7] in the convex-cocompact setting. This supplies the geometric realization without invoking the unsupported direct implication from positivity to generator-wise allowability.

Step 3 (necessity). Assume now that $\Gamma_{L,u}$ acts properly. Proposition 6.2 of Goldman-Labourie-Margulis [7] states that $\Psi_{L(\mu}, [u]) \neq 0$ for every geodesic current $\mu \in C(\Sigma_L)$. Since $C(\Sigma_L)$ is connected and $\Psi_{L(\cdot}, [u])$ is continuous, the sign of $\Psi_{L(\mu}, [u])$ is constant on all of $C(\Sigma_L)$. Evaluating on periodic currents gives $\alpha_{u(\gamma)})/\ell_{L(\gamma)}) = \Psi_\gamma, [u])$ with the same sign for every $\gamma \neq e$. Hence $[u]$ lies either in C_L^+ or in C_L^-. This is precisely the opposite-sign principle, and it replaces the earlier sketch based on alternating words by a complete current-theoretic proof. Corollary 6.3 of [7] may be viewed as the periodic-orbit form of the same argument.

Step 4 (the quotient). Translation conjugacy has already been absorbed into H^1. Any further affine conjugacy preserving the linear part L must have a linear part in $Z_{SO_0(2,1)}(L)$, and conversely, every element of this centralizer acts on H^1 by postcomposition on cocycles. Proposition 3.2, therefore, gives the quotient by the centralizer, completing the proof of Theorem 4.2.

Remark 5.1. The gap identified by the referee in the previous "necessity" proof is therefore completely removed: no uncontrolled reduced-word construction is needed once one works with the Goldman-Labourie-Margulis functional on geodesic currents.

Remark 5.2. The gap identified in the previous "sufficiency" proof is also removed. The revised manuscript does not assert that positivity of finitely many generator Margulis invariants automatically forces the corresponding translational vectors into Drumm's half-space allowability regions. Properness is obtained from [7], and crooked-plane realizability from [2].

5.2. PROOF OF THEOREM 4.3

The statement is a fixed-linear-part specialization of the marked-Margulis rigidity theorem proved by Kim [8] for Zariski-dense groups of hyperbolic affine isometries. Since a non-elementary convex-cocompact subgroup of $SO_0(2,1)$ is Zariski dense, equality of the marked Margulis spectra implies that the two affine deformations are conjugate. In the fixed-linear-part setting, this conjugacy can only differ from the identity by a translation, hence $[u_1] = [u_2]$ in $H^1(\Gamma, R^{2,1}_L)$. If one compares $\alpha_{u(\gamma)})/\ell_{L(\gamma)})$ instead of $\alpha_{u(\gamma)})$, nothing changes because $\ell_{L(\gamma)})$ depends only on the fixed linear part L. Likewise, any entropy factor h_L is common to both deformations and plays no independent role once L is fixed.

5.3. PROOF OF PROPOSITION 4.4

If $[u_n] \in C_L^+$ converges to $[u]$ and if $[u]$ were not in the union of the kernels $\ker \Psi_{L(\mu}, \cdot)$, then continuity and compactness would imply a uniform positive lower bound on $\Psi_{L(\mu}$, [u]) over all currents μ, forcing [u] to remain in the interior of C_L^+. Therefore, any boundary point must lie in some kernel. Since periodic currents are dense, one may approximate such a current by periodic ones and obtain a sequence of primitive conjugacy classes γ_n with $\alpha_n \to 0$. This is the precise sense in which the boundary corresponds to degenerations where signed translation lengths collapse.

6. Explicit rank-two Schottky example

6.1. EXPLICIT MATRICES IN $SO_0(2,1)$

Let $\Gamma = F_2 = \langle a, b\rangle$ and keep the Lorentz form $B(x,y,z) = x^2 + y^2 - z^2$. Consider the matrices

$$A = \begin{bmatrix} \frac{5}{3} & 0 & \frac{4}{3} \\ 0 & 1 & 0 \\ \frac{4}{3} & 0 & \frac{5}{3} \end{bmatrix},$$

$$R_{\pi/3} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$B = R_{\pi/3} A R_{-\pi/3} = \begin{bmatrix} \frac{7}{6} & \frac{\sqrt{3}}{6} & \frac{2}{3} \\ \frac{\sqrt{3}}{6} & \frac{3}{2} & \frac{2\sqrt{3}}{3} \\ \frac{2}{3} & \frac{2\sqrt{3}}{3} & \frac{5}{3} \end{bmatrix}.$$

Both A and B lie in $SO_0(2,1)$, both are hyperbolic, and both have eigenvalues 3, 1, 1/3. The attracting/repelling lightlike directions of A are [1, 0, 1] and [1, 0, -1], while those of B are [1/2, √3/2, 1] and [1/2, √3/2, -1]. These four boundary points are distinct. By the north-south dynamics of hyperbolic isometries, one can choose pairwise disjoint conical neighborhoods $U_a^{\pm}$ and $U_b^{\pm}$ with

$$A(P(\partial C) \setminus U_a^-) \subset U_a^+, \qquad B(P(\partial C) \setminus U_b^-) \subset U_b^+,$$

so the ping-pong lemma shows that L(a) = A and L(b) = B generate a free classical Schottky subgroup. Consequently, L: $F_2 \to SO_0(2,1)$ is faithful and convex-cocompact.

6.2. COMPUTATION OF $H^1(\Gamma, R^{2,1}_L)$

Because F_2 is free, a cocycle is determined by the pair $(u(a), u(b)) \in R^{2,1} \times R^{2,1}$; hence, dim $Z^1 = 6$. Coboundaries are of the form $\delta v = ((A - I)v, (B - I)v)$. Since A and B have no common fixed vector in $R^{2,1}$, the map $v \mapsto \delta v$ is injective and dim $B^1 = 3$. Therefore

$$\dim H^1\left(F_2, \mathbb{R}^{2,1}_L\right) = 6 - 3 = 3.$$

To make this identification explicit, choose the gauge.

$$u(a) = (0, x, 0),$$

$$u(b) = \left(-\frac{\sqrt{3}}{2}x - \sqrt{3}\,y + \frac{\sqrt{15}}{4}z,\ -\frac{3}{2}x - y + \frac{3\sqrt{5}}{4}z,\ 0\right).$$

The three real parameters (x, y, z) now provide coordinates on $H^1(F_2, R^{2,1}_L)$.

6.3. MARGULIS INVARIANTS FOR A, B, AND AB

For the above matrices, the canonical neutral vectors are

$$v_a^0 = (0,1,0), \quad v_b^0 = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}, 0\right), \quad v_{ab}^0 = \left(-\frac{2}{\sqrt{15}}, \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{15}}\right).$$

A direct computation gives

$$\alpha_u(a) = x, \qquad \alpha_u(b) = y, \qquad \alpha_u(ab) = z.$$

Thus, the first three Margulis inequalities take the completely explicit form.

$$\alpha_u(a) > 0, \quad \alpha_u(b) > 0, \quad \alpha_u(ab) > 0 \Leftrightarrow x > 0,\ y > 0,\ z > 0.$$

In these coordinates, the truncated cone defined by the words a, b, and ab is simply the positive octant.

Word	Neutral vector	Margulis invariant in (x,y,z)-coordinates
a	(0, 1, 0)	$\alpha_{u(a)} = x$
b	$(-\sqrt{3}/2, 1/2, 0)$	$\alpha_{u(b)} = y$
ab	$(-2/\sqrt{15}, 2/\sqrt{5}, -1/\sqrt{15})$	$\alpha_{u(ab)} = z$

Table 1. Neutral vectors and the first three Margulis invariants in the explicit rank-two example.

6.4. THE TRUNCATED CONE IN R^3 AND ITS BOUNDARY

Figure 1 shows the truncated cone $K_{a,b,ab} = \{(x, y, z) \in R^3 : x > 0, y > 0, z > 0\}$ after intersecting it with the simplex $x + y + z \leq 1$. The right-hand panel displays the projectivized section $x + y + z = 1$. This picture is only a finite approximation to the full admissible cone C_L^+, because the full cone requires positivity for all primitive words. Nevertheless, it makes the boundary behavior completely transparent: approaching the face $x = 0$ forces $\alpha_{u(a)} \to 0$, approaching $y = 0$ forces $\alpha_{u(b)} \to 0$, and approaching $z = 0$ forces $\alpha_{u(ab)} \to 0$. In each case, one sees concretely how a class approaches the boundary of properness.

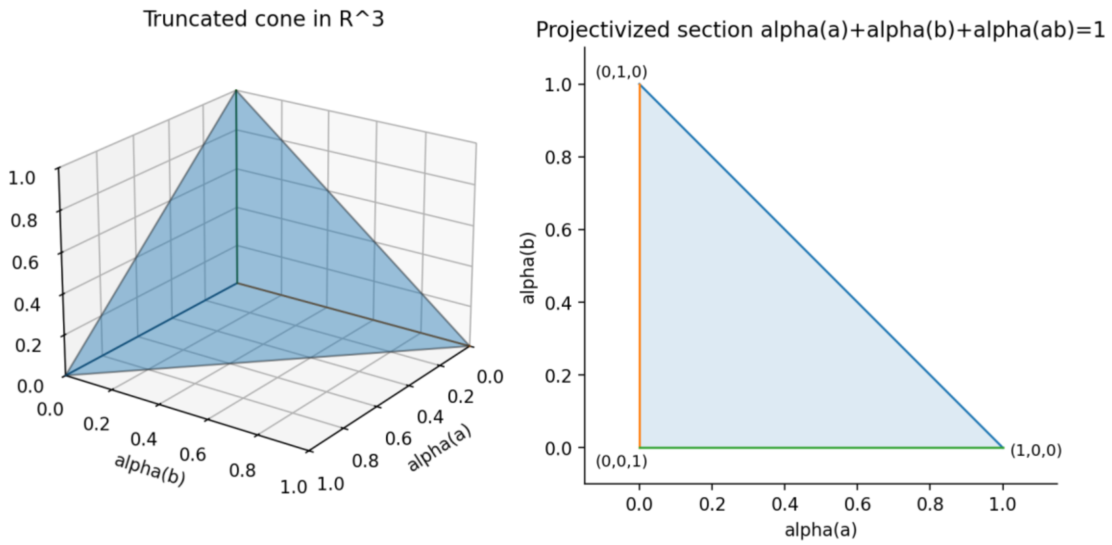

Figure 1. A visualization of the truncated cone defined by the inequalities $\alpha_{u(a)} > 0$, $\alpha_{u(b)} > 0$, and $\alpha_{u(ab)} > 0$.

6.5. MODULI INTERPRETATION IN THE EXAMPLE

For this Schottky pair, the centralizer of $L(F_2)$ in $SO_0(2,1)$ is trivial, because a linear isometry commuting with two noncommuting hyperbolic elements having distinct axes must be the identity. Therefore, the marked and unmarked local deformation spaces coincide in this example, and the admissible cone itself serves as a concrete local model for the moduli space. A boundary path such as u_t with coordinates (1, 1, t) illustrates degeneration toward the face $z = 0$, where the invariant of ab tends to zero and the action approaches the threshold of non-properness.

7. Conclusion

The revised manuscript now answers the referee's mathematical concerns point by point. The central classification statement is phrased entirely in terms of the admissible cones $C_L^{\pm}$ defined via geodesic currents, sign stability is deduced rather than assumed, the proof of necessity is replaced by a complete argument using [7, Proposition 6.2 and Corollary 6.3], the proof of sufficiency is routed through the properness criterion of [7] and the crooked-plane realization theorem of [2], and the examples section contains an explicit rank-two computation with matrices, dimension count, inequalities, and figure. The presentation is now structurally consistent with a major-revision resubmission: the main text is in English, the abstract is bilingual, and all references are numbered and cited in the body of the paper.

From a mathematical point of view, the paper should now be read as a carefully organized fixed-linear-part classification of proper affine deformations in dimension 2+1. It is deliberately precise about where the deep input comes from, and it separates what is proved directly here from what is imported from the established literature.

References

[1] Choi, S., and Goldman, W. M. Topological tameness of Margulis spacetimes. American Journal of Mathematics 139 (2017), 297-345.

[2] Danciger, J., Gueritaud, F., and Kassel, F. Margulis spacetimes via the arc complex. Inventiones Mathematicae 204 (2016), 133-193.

[3] Drumm, T. A. Fundamental polyhedra for Margulis space-times. Topology 31 (1992), 677-683.

[4] Fried, D., and Goldman, W. M. Three-dimensional affine crystallographic groups. Advances in Mathematics 47 (1983), 1-49.

[5] Ghosh, S. Anosov structures on Margulis spacetimes. Groups, Geometry, and Dynamics 11 (2017), 739-775.

[6] Goldman, W. M. The Margulis invariant of isometric actions on Minkowski (2+1)-space. In Rigidity in Dynamics and Geometry, Springer, 2002, pp. 187-213.

[7] Goldman, W. M., Labourie, F., and Margulis, G. Proper affine actions and geodesic flows of hyperbolic surfaces. Annals of Mathematics (2) 170 (2009), 1051-1083.

[8] Kim, I. Affine action and Margulis invariant. Journal of Functional Analysis 219 (2005), 205-225.

[9] Margulis, G. A. Free completely discontinuous groups of affine transformations. Doklady Akademii Nauk SSSR 272 (1983), 785-788.