JOBS
مجلة العلوم الأساسية
Journal of Basic Science
Print -ISSN 2306-5249
Online-ISSN 2791-3279
العدد السابع والاربعون
٢٠٢٦م /١٤٤٨هـ

(593) (575)
العدد السابع والاربعون

حلول عددية باستخدام مصفوفة تشغيلية لمعادلات تفاضلية كسرية غير منفردة ذات رتبة متغيرة

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المستخلص:

لحل المعادلات التفاضلية الكسرية ذات الرتبة المتغيرة والتأخير المتغير التي تتضمن مشتقة أتانجانا–بالينو، يُقترح استخدام طريقة التجميع الطيفي بالاعتماد على مصفوفة ليجندر التشغيلية المُزاحة. ولحساب حد التأخير المتغير، يتم إنشاء مصفوفة ليجندر تشغيلية جديدة بالاستفادة من خصائص كثيرات حدود ليجندر المُعَيَّرة، إلى جانب مصفوفة تشغيلية مقابلة للمشتقة الكسرية لكثيرات حدود ليجندر. كما تُستخدم هذه المصفوفات التشغيلية لتقدير حدود خطأ تقريب المشتقة الكسرية. وبالاعتماد على نهج التجميع القائم على هذه المصفوفات التشغيلية، تُحوَّل المعادلات التفاضلية الكسرية ذات الرتبة المتغيرة والتأخير المتغير، والمتضمنة لمشتقة أتانجانا–بالينو، إلى نظام من المعادلات الجبرية. وبالمقارنة مع الطرق السابقة، تؤكد النتائج العددية دقة وكفاءة الطريقة المقترحة، مما يبرز قدرتها على توفير أداة عددية فعالة لمعالجة المعادلات التفاضلية الكسرية ذات الرتبة المتغيرة والتأخير المتغير التي تتضمن مشتقة أتانجانا–بالينو.

الكلمات المفتاحية:

طريقة التجميع الطيفي؛ الرتبة المتغيرة؛ المعادلات التفاضلية الكسرية؛ مشتقة أتانجانا–بالينو؛ النواة غير المفردة

JOBS Journal of Basic Science مجلة العلوم الأساسية
Print -ISSN 2306-5249
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٢٠٢٦م /١٤٤٨هـ

NUMERICAL SOLUTIONS USING AN OPERATIONAL MATRIX FOR NON-SINGULAR KERNEL FRACTIONAL DELAY DIFFERENTIAL EQUATIONS WITH VARIABLE-ORDER

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Abstract

To solve variable-order fractal-fractional delay equations involving the Atangana-Baleanu derivative, an optimal collocation technique using a shifted Legendre operational matrix is proposed. To address these problems, we created a new Legendre varying-delay operational matrix. Normalised Legendre polynomial features, together with a corresponding Legendre fractional derivative operational matrix, are used in this method. This also displays the fractional derivative error bound's Normalized Legendre operational matrix. A collocation approach based on these operational matrices reduces the variable-order fractal-fractional varying-delay differential equations with Atangana-Baleanu derivatives to a set of algebraic equations. When compared to previous methods, the numerical results validate the effectiveness of the suggested method as a robust mathematical tool, outperforming existing techniques in handling variable-order fractal-fractional varying-delay differential equations with an Atangana-Baleanu derivative.

Keywords: Variable-order Spectral method; Fractal differential equations; Nonsingular kernel derivative

1. Introduction

In the domains of science and engineering, fractional calculus offers a crucial approach for examining arbitrary-order derivatives and integrals. When examining intricate physical processes that defy conventional integer-order modeling, this method works especially well (Aiello et al., 1992). These equations have numerous applications in the applied sciences; for example, numerous researchers have studied physics, biology,

JOBS مجلة العلوم الأساسية
Journal of Basic Science
Print -ISSN 2306-5249
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electrodynamics, and economics (Al-Sudani et al., 2024; Atangana et al., 2016). Heterogeneity is efficiently expressed by fractional differential and integral operators, particularly those that follow both for earlier and later times and those that adhere to natural rules like the power law and exponential decay law (Basim et al., 2022; Kumar et al.,2019). Mathematicians have developed various mathematical operators, such as the Atangana–Baleanu Caputo (ABC) fractional differential and integral operators, Riemann–Liouville, and Caputo–Fabrizio.

Variable-order differential operators are being used more and more by researchers to address challenging real-world issues that are outside the scope of conventional approaches. The operational matrices for fractional integration and differentiation are fundamental tools for solving numerical solutions to fractional differential equations, as established in (Kumar et al., 2020). In this study, we will utilize legendre polynomials, which are a kind of Appell polynomial. This polynomial exhibit highly accurate operational matrices of derivatives, even though it is not based on orthogonal functions. It is important to note that, in terms of generic function estimation, this polynomial offers significant advantages over standard alternative orthogonal polynomials, just like the Jacobi and Legendre polynomials, the reader can see (Veeresha et al., 2020) for further details on these benefits.

In this article, the operational matrix method (OMM) will be used, incorporated with normalized shifted Legendre polynomials (NSLPs) in order to solve variable-order fractal fractional Atangana–Baleanu Caputo (VOFFABC) featuring varying delay differential equations. The fundamental theorem of fractional calculus (Veeresha et al., 2020) and a two-step Lagrange polynomial will be compared. Complex differential problems can be reduced to solvable systems of algebraic equations with remarkable precision using the (OMM).

In contrast to other methods, it is carried out to suitably illustrate the simplicity and applicability of convergent results. The alternate method requires more subroutines to get accurate results, which complicates and lengthens the computation.

The proposed work offers several interesting benefits. The main advantage is the method for exhibiting the operational matrix of integration, which is accomplished very precisely. This characteristic reflects the computational approach, and as a result, we get an estimate that is extremely near to the precise solution.

This paper is divided into five sections: The definitions of fractal fractional derivatives are emphasized in Section 2. Section 3 will provide the OMM for VOFFABC delay differential equations. Numerous numerical examples and solutions are included in Section 4. There are some conclusions in the last part.

1. Mathematical Concepts

This section investigates the application of fractal fractional derivatives Utilizing the Mittag-Leffler kernel.

Definition 1.1. Consider $0 < \sigma < 1$, $q(\psi) \in H^1(a,b), b > a$. The ABC fractional derivative's mathematical expression can be represented in (Kumar et al,.2019)

$$ {}_{0}^{ABC}G^{\sigma} q(\psi) = \frac{M(\sigma)}{1-\sigma} \int_{0}^{\psi} q'(s) E_{\sigma} \left[\frac{-\sigma(\psi-u)^{\sigma}}{1-\sigma} \right] du, \tag{1} $$

where $M(\sigma)$ stands for a normalization function $0 < \sigma < 1$, while E_{σ} denotes the Mittag-Leffler function.

To expand the ABC-derivative definition in the scenario where $n < \sigma < n+1$ having $q^{(u)}(a) = 0$, $u = 1,2,\ldots,n$, we have:

$$ {}_{0}^{ABC}G^{\sigma} q(\psi) = {}^{ABC}G^{\sigma}(G^{n} q(\psi)) $$

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مجلة العلوم الأساسية
Print -ISSN 2306-5249
Online-ISSN 2791-3279
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$$= \frac{M(\sigma)}{1-\sigma} \int_0^{\psi} q^{(n+1)}(u) E_\sigma \left[\frac{-\sigma}{1-\sigma} (\psi - u)^\sigma \right] du$$

$$q^{(n+1)}(\psi) = G^{(n+1)} q(\psi) = G^{\lceil \sigma \rceil} q(\psi). \qquad (2)$$

in which $\lceil \sigma \rceil$ indicates the ceiling σ.

Definition 1.2. One way to express the fractional integral that corresponds to the ABC-derivative is as

$$ {}_0^{AB}I_\psi^\sigma \{q(\psi)\} = \frac{1-\sigma}{M(\sigma)} q(\psi) + \frac{\sigma}{M(\sigma)\Gamma(\sigma)} \int_0^{\psi} q(u)(\psi - u)^{\sigma - 1} du. \qquad (3)$$

Definition 1.3. The ABC-derivative with variable-order $\sigma(\psi), 0 < \sigma(\psi) < 1$ of the function $q(\psi)$ can be written as (Kumar et al. 2020):

$$ {}_0^{ABC}G^{\sigma(\psi)} q(\psi) = \frac{M(\sigma(\psi))}{1-\sigma(\psi)} \int_0^{\psi} q'(u) E_{\sigma(\psi)} \left[\frac{-\sigma(\psi)}{1-\sigma(\psi)} (\psi - u)^{\sigma(\psi)} \right] du. \qquad (4)$$

where $E_{\sigma(\psi)}(\psi)$ similar to the Mittag-Leffler function.

Definition 1.4. Let the fractal differentiable and continuous mapping on the interval (a,b) with order β be represented by the function $q(\psi)$. Consequently, the ψ function of FFABC-derivative of order σ is illustrated as follows (Veeresha et al., 2020):

$$ {}^{FFABC}_{0}G^{\sigma,\beta} q(\psi) = \frac{AB(\sigma)}{(1-\sigma)} \int_0^{\psi} \frac{dq(u)}{du^{\beta}} E_{\sigma}\left[\frac{-\sigma}{1-\sigma}(\psi-u)^{\sigma}\right] du, \tag{5} $$

$$ 0 < \sigma, \beta \le 1, \frac{dq(u)}{du^{\beta}} = \lim_{\psi \to u} \frac{q(\psi)-q(u)}{\psi^{\beta}-u^{\beta}}. $$

The integral possesses a nonsingular and nonlocal Mittag-Leffler kernel, where $E\sigma$ represents a one-parameter Mittag-Leffler function as follows:

$$ E_{\sigma}(-\psi^{\sigma}) = \sum_{k=0}^{\infty} \frac{(-\psi)^{\sigma k}}{\Gamma(\sigma k+1)} $$

and

$$ AB(\sigma) = 1 + \frac{\sigma}{\Gamma(\sigma)} - \sigma $$

For order $\sigma(\psi)$ and $\beta(\psi)$, we can get:

$$ {}^{VOFFABC}G^{\sigma(\psi),\beta(\psi)} q(\psi) = \frac{AB(\sigma(\psi))}{(1-\sigma(\psi))} \int_0^{\psi} \frac{dq(u)}{du^{\beta(\psi)}} E_{\sigma(\psi)}\left[\frac{-\sigma(\psi)}{1-\sigma(\psi)}(\psi - u)^{\sigma(\psi)}\right] du. \tag{6} $$

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Journal of Basic Science
Print -ISSN 2306-5249
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1.1 Several properties of NSLPs

The analytic form of the NSLP $N\iota(\psi)$ of degree ι defined on $[0,1]$, is given by:

$$N_{\iota}(\psi)=\sqrt{2\iota+1}\sum_{k=0}^{\iota}(-1)^{\iota-k}\frac{(\iota+k)!}{(\iota-k)!}\frac{\psi^{k}}{(k!)^{2}}.$$

In terms of NSLP, a function $q(\psi)$ that is square integrable in $[0,1]$ may be written as:

$$q(\psi)=\sum_{j=0}^{\infty}z_{j}\hbar_{j}(\psi),$$

where the coefficients z_j can be expressed as follows:

$$z_{j}=\sqrt{2j+1}\int_{0}^{1}q(\psi)\hbar_{j}(\psi)d\psi,\qquad j=1,2,\ldots.$$

In the solution, only the first $(m+1)-$term NSLP are taken into account. Therefore, we have:

$$q(\psi)=\sum_{j=0}^{m}z_{j}\hbar_{j}(\psi)=Z^{T}\varphi(\psi),$$

In the following equation, Z stands for the shifted Legendre coefficient vector and $\phi(\psi)$ for the shifted Legendre vector:

$$Z^{T}=[z_{0},z_{1},\ldots,z_{m}],$$
$$\varphi(\psi)=[N_{0}(\psi),N_{1}(\psi),\ldots,N_{m}(\psi)]^{T}. \qquad (7)$$

JOBS مجلة العلوم الأساسية Journal of Basic Science
Print -ISSN 2306-5249
Online-ISSN 2791-3279
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The vector $\varphi(\psi)$ derivative can be expressed as follows:

$$\frac{d\varphi(\psi)}{d\psi} = D^{(1)}\varphi(\psi),$$

where $G(1)$ is the derivative's $(m + 1) \times (m + 1)$ OM, as determined by:

$$\wp^{(1)} = d_{\iota j}$$
$$= \begin{cases} 2\sqrt{2l+1}\sqrt{2j+1} & for \;\; j = \iota - k, \begin{cases} k = 1,3,\dots,m & if \;\; m \;\; odd \\ k = 1,3,\dots,m-1 & if \;\; m \;\; even \end{cases} \\ 0, & otherwise \end{cases}$$

It is obvious that

$$\frac{d^n\varphi(\psi)}{d\psi^n} = (G^{(1)})^n\varphi(\psi),$$

where matrix powers are indicated by $n \in N$ and the superscript in $G^{(1)}$. As a result,

$$G^{(n)} = (G^{(1)})^n, n = 1,2, \dots$$

2.1 Delay operational matrix

We define the delay-shifted Legendre orthonormal vector $q(\psi - \zeta(\psi))$ or $q(\zeta(\psi)\psi)$ as follows:

$$q(\psi - \zeta(\psi)) = R\phi(\psi). \tag{8}$$

In this context, the (R) analogous to the $N \times N$ operational delay matrix represented by:

$$R = A_1 A^{-1},$$

where $A_1 = \int_0^1 \phi(\psi - \zeta(\psi))\phi^T(\psi)d\tau$ and $A = \int_0^1 \phi(\psi)\phi^T(\psi)d\psi$. Additionally, any delay function $q(\psi\zeta(\psi))$ or $q(\zeta\psi)$ can be defined in terms of shifted Legendre polynomials as follows:

$$q(\psi - \zeta(\psi)) = Z^T R\phi(\psi). \tag{9}$$

where Z is given in eq.(11).

2. OMM of VOFFABC delay differential equations

The numerical solution of the issue using the OMM of VOFFABC delay differential eq. [13,14] is described in this part and yields the following results:

$$VOFFABC_G^{\sigma(\psi),\beta(\psi)} q(\psi) = \beta(\psi)\psi^{\beta(\psi)-1} f(\psi, q) \tag{1٠}$$

for

$$\phi(\psi) = [P_0(\psi), P_1(\psi), \dots, P_m(\psi)]^T,$$
$$\theta(\psi) = [1, \psi, \psi^2, \dots, \psi^n]^T. \tag{1١}$$

Consequently, the vector $\phi(\psi)$ can be represented as:

$$\phi(\psi) = A\theta(\psi), \tag{1٢}$$

where A is a square matrix $(\chi + 1) \times (\chi + 1)$ expressed as:

$$(a_{i,j})_{0\le i,j\le n} = \begin{cases} (-1)^{i-j} \dfrac{\Gamma(i+1)\Gamma(i+j+1)}{\Gamma(j+1)\Gamma(i+1)\Gamma(i-j+1)\Gamma(j+1)} & for \quad i \ge j, \\ 0, & for \quad otherwise \end{cases} \quad (13)$$

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Journal of Basic Science
Print -ISSN 2306-5249
Online-ISSN 2791-3279
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Consequently, using eq. (12), we state that

$$\phi(\psi) = A^{-1}\theta(\psi), \tag{14}$$

where the OMM is functionally dependent on the SLPs linked to the VOFFABC operator $^{VOFFABC}G^{\sigma(\psi)}\phi(\psi)$. By using eq.(14) we achieve:

$$^{VOFFABC}G^{\sigma(\psi),\beta(\psi)}\phi(\psi) \quad ^{VOFFABC}G^{\sigma(\psi),\beta(\psi)}\left(A\lambda(\psi)\right) \tag{15}$$

$$= A^{VOFFABC}G^{\sigma(\psi),\beta(\psi)}[1,\psi,\psi^2,\dots,\psi^n]^T.$$

In this instance, the ABC-derivative of the variable-order given in eq. (4) may be applied. Consequently, eq. (15) can be formulated in the following step:

$$^{VOFFABC}G^{\sigma(\psi),\beta(\psi)}\,\phi(\psi) = \left[0, \frac{B\big(\sigma(\psi)\big)\Gamma(2)}{\Gamma\big(1-\sigma(\psi)\big)}\psi\sum_{k=0}^{\infty}\frac{\left(\frac{-\sigma(\psi)}{1-\sigma(\psi)}\psi^{\sigma(\psi)}\right)^k}{\Gamma(k\sigma(\psi)+2)}, \frac{B\big(\sigma(\psi)\big)\Gamma(3)}{\Gamma\big(1-\sigma(\psi)\big)}\psi^2\sum_{k=0}^{\infty}\frac{\left(\frac{-\sigma(\psi)}{1-\sigma(\psi)}\psi^{\sigma(\psi)}\right)^k}{\Gamma(k\sigma(\psi)+3)}, \dots\right.$$

$$\left.\sum_{k=0}^{\infty}\frac{B(\sigma(\psi))\Gamma(n+1)}{\Gamma(1-\sigma(\psi))}\psi^n\sum_{k=0}^{\infty}\frac{\left(\frac{-\sigma(\psi)}{1-\sigma(\psi)}\psi^{\sigma(\psi)}\right)^k}{\Gamma(k\sigma(\psi)+n+1)}\right]^T$$

$$= A$$

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{B(\sigma(\psi))\Gamma(2)}{\Gamma(1-\sigma(\psi))}\sum_{k=0}^{\infty}\frac{(\frac{-\sigma(\psi)}{1-\sigma(\tau)}\psi^{\sigma(\psi)})^k}{\Gamma(k\sigma(\psi)+2)} & 0 & \cdots & 0 \\ 0 & 0 & \frac{B(\sigma(\psi))\Gamma(3)}{\Gamma(1-\sigma(\psi))}\sum_{k=0}^{\infty}\frac{(\frac{-\sigma(\psi)}{1-\sigma(\tau)}\psi^{\sigma(\psi)})^k}{\Gamma(k\sigma(\psi)+3)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{B(\sigma(\psi))\Gamma(n+1)}{\Gamma(1-\sigma(\psi))}\sum_{k=0}^{\infty}\frac{(\frac{-\sigma(\psi)}{1-\sigma(\tau)}\psi^{\sigma(\psi)})^k}{\Gamma(k\sigma(\psi)+n+1)} \end{pmatrix}\begin{pmatrix} 1 \\ \psi \\ \psi^2 \\ \vdots \\ \psi^n \end{pmatrix}$$

$$= A\ \ B(\psi)\theta(\psi) \tag{16}$$

Substuting eq.(16) in eq.(15), we get

$$^{VOFFABC}G^{\sigma(\psi)}\ \phi(\psi) = AB(\psi)A^{-1}\theta(\psi) \tag{17}$$

The approximate solution can be presented

$$^{VOFFABC}G^{\sigma(\psi),\beta(\psi)}q(\psi) \simeq\ ^{VOFFABC}G^{\sigma(\psi),\beta(\psi)}(Z^T\phi(\psi)) = Z^T\ ^{VOFFABC}G^{\sigma(\psi),\beta(\psi)}\phi(\psi) = Z^TAB(\psi)A^{-1}\phi(\psi). \tag{18}$$

$$\begin{gathered} Z^TAB(\psi)A^{-1}\phi(\psi) = \Gamma[\psi, Z^T\phi(\psi)Z^TR\phi(\psi), \dots], \\ 0 \le \psi \le 1. \end{gathered} \tag{19}$$

In this instance, the system of equations in eq. (19) is converted into a similar system of algebraic equations by using collocation points, $\psi_i = \frac{2i+1}{2n+2}$, for $i\ =\ 0, 1, 2, \dots, n$, given as follows:

$$\begin{gathered} Z^TAB(\psi_i)A^{-1}\phi(\psi_i) = \Gamma[\psi_i, Z^T\phi(\psi_i)Z^TR\phi(\psi_i), \dots], \\ Z^T\phi(\psi_i) = \psi_0. \end{gathered} \tag{20}$$

Conclusively, the system of algebraic equations given in eq.(20) can

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Journal of Basic Science
Print -ISSN 2306-5249
Online-ISSN 2791-3279
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be solved to find the unknown vector Z from eq. (11).

3. Numerical Examples

In the computational results, the accuracy of the approximate solution relative to the exact solutions will be measured by the absolute error.

- **LOM** Legendre Operational matrix method derived in this study.
- **PRCO** Predictor-Corrector method (Basim, M.et al, 2026).
- **MTSLP** a mixture of two-step Lagrange polynomial as well as the fundamental theorem of fractional calculus (Toufik et al, 2017).

Example 1. Suppose the ABC sense of the variable-order fractional delay differential eq. (Kumar et al,.2020):

$$G^{\sigma(\psi)}q(\psi) = q(\psi-\zeta(\psi)) - q(\psi) + f(\psi), \psi \in [0,1], \sigma(\psi) \in [0,1]$$

$$f(\psi) = 2\zeta(\psi)\psi - \zeta(\psi) - \zeta^2(\psi) + \frac{2AB(\sigma(\psi))\psi^2}{1-\sigma(\psi)} E_{\sigma(\psi),3}\left(\frac{-\sigma(\psi)}{1-\sigma(\psi)}\psi^{\sigma(\psi)}\right) - \frac{AB(\sigma(\psi))\psi}{1-\sigma(\psi)} E_{\sigma(\psi),2}\left(\frac{-\sigma(\psi)}{1-\sigma(\psi)}\psi^{\sigma(\psi)}\right),$$

where $q(\psi) = 0, \psi \leq 0$. The exact solution $q(\psi) = \psi^2 - \psi$.

For $\sigma(\psi) = tanh(\psi+1), \zeta(\psi) = 0.001$ and $m = 8$ To demonstrate the precision of the computational algorithm, show the results obtained from the following in Table (1) and Figure (1).

Example 2. Consider the VOFABC delay differential equation, presented by (Veeresha et al,. 2020):

$$G^{\sigma(\psi)}q(\psi) = -q(\psi) + \frac{\zeta(\psi)}{2q\left(\frac{\zeta(\psi)}{2\psi}\right)} - \frac{\zeta(\psi)}{2e^{-\zeta(\psi)\psi}},$$

$$q(0) = 1, \qquad \psi \in [0,1], \qquad \sigma(\psi) \in [0,1].$$

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Print -ISSN 2306-5249
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Where $\sigma(\psi) = 10.06\,cos(\psi)\,sin(\psi)$, and $m = 8$ the exact solution in case $\sigma(\psi) = 1$ is $q(\psi) = e - \psi$. An approximate solution is shown in Table (2), and Figure (2).

Table 1: Numerical comparison of the exact and approximate solutions for Example (1) using alternative approaches.

ψ	Exact	LOM	PRCO	MTSLP
0.1	-0.09000	-0.08999	-0.09048	-0.05538
0.2	-0.16000	-0.15999	-0.16030	-0.13345
0.3	-0.21000	-0.21000	-0.20896	-0.21063
0.4	-0.24000	-0.24000	-0.23732	-0.25255
0.5	-0.25000	-0.24999	-0.24573	-0.26531
0.6	-0.24000	-0.24000	-0.23426	-0.25353
0.7	-0.21000	-0.21000	-0.20280	-0.21997
0.8	-0.16000	-0.15999	-0.15132	-0.16589
0.9	-0.09000	-0.09000	-0.08857	-0.09055
Time		85.23400	11.75000	6.25999

Table 2: Numerical comparison of the exact and approximate solutions for Example (2) with other methods.

ψ	Exact	LOM	PRCO	MTSLP
0.1	0.90483	0.90478	0.94378	0.95988
0.2	0.81873	0.81943	0.84788	0.85669
0.3	0.74081	0.74324	0.76225	0.77184
0.4	0.67032	0.67531	0.68625	0.69651
0.5	0.60653	0.61469	0.61898	0.63094

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0.6	0.54881	0.56047	0.55934	0.57316
0.7	0.49658	0.51178	0.50624	0.52186
0.8	0.44932	0.46788	0.45858	0.47565
0.9	0.40656	0.42811	0.41537	0.43336
Time		5.25000	3.86999	4.64000

Table 3: An approximate solution by using the LOM method for Example 2 with different m.

ψ	$m = 4$	$m = 6$	$m = 8$	Exact
0.1	-0.08999	-0.09000	-0.09000	-0.09000
0.2	-0.15999	-0.15999	-0.16000	-0.16000
0.3	-0.20999	-0.20999	-0.21000	-0.21000
0.4	-0.23999	-0.24000	-0.24000	-0.24000
0.5	-0.24999	-0.25000	-0.25000	-0.25000
0.6	-0.23999	-0.24000	-0.24000	-0.24000
0.7	-0.20999	-0.20999	-0.21000	-0.21000
0.8	-0.15999	-0.16000	-0.16000	-0.16000
0.9	-0.09000	-0.09000	-0.09000	-0.09000

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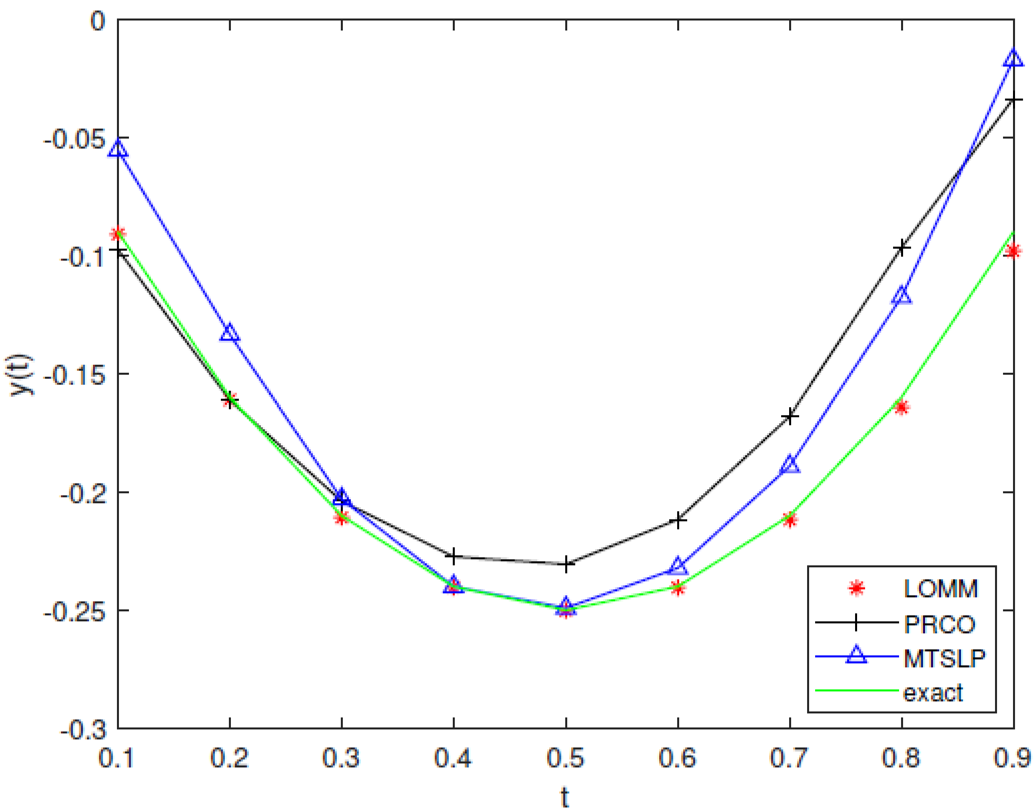

Fig. 1: The comparison of the exact solution and the approximation solution methods

when $\zeta(\psi) = 0.001e - \psi$ in Example (1)

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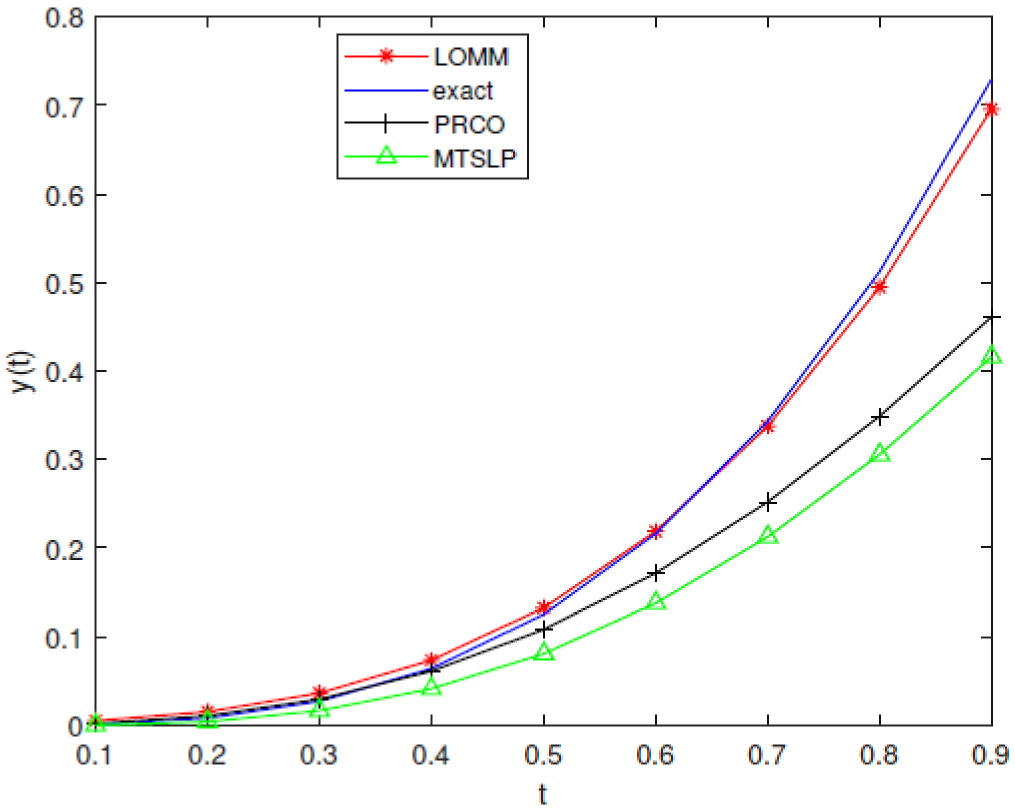

Fig. 2: The comparison of the exact solution and the approximation solution methods in Example (2).

4. Conclusions

In this work, we introduced an operational matrix based on SLPs for VOFFABC time-delay equations. This derivative's recent development has created a new framework for differential equations. Several case studies were examined and resolved in order to show the efficacy and accuracy of the suggested techniques. Additionally, this study seeks to compare the outcomes with well-established numerical techniques. The suggested method outperforms current alternatives in terms of computational efficiency and dependability, according to a comparative study. Through the incorporation of memory effects and fractal features into the model frameworks, numerical simulations uncover previously

JOBS مجلة العلوم الأساسية
Journal of Basic Science
Print -ISSN 2306-5249
Online-ISSN 2791-3279
العدد السابع والاربعون
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hidden unique dynamical behaviors. While the systems exhibit stable periodic orbits across various variable orders, alternative values trigger a transition into richer and more diverse dynamics. The proposed method can be applicable to a broader class of biological systems; it has been successfully applied in other technical domains such as engineering, finance, and economics, as well as in the modeling of infectious diseases. We are confident that this essay will provide many fresh avenues for researching the modeling of real-world scenarios.

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