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نماذج العمليات الغاوسية البايزية لانحدار كونواي–ماكسويل–بواسون المرن

باسل فريح محمد الحرداني
التخصص إحصاء رياضي
baslfryh8@gmail.com

المستخلص:

في العلوم الكمية، تُعد نماذج الانحدار لبيانات العد أساسية. كما أن الافتراضات التوزيعية الصارمة تحد من الطرق القياسية. فبينما تعالج حلول مثل انحدار ذي الحدين السالب مشكلة التشتت الزائد، إلا أنها لا تأخذ في الحسبان مشكلة التشتت الناقص، وهو انتهاك شائع آخر لافتراض تساوي التشتت في نموذج بواسون. يوفر توزيع كونواي–ماكسويل–بواسون (COM–Poisson) تعميمًا قويًا يمكنه نمذجة كلتا الظاهرتين. ومن عيوبه الهامة الأخرى أنه يُطبق ضمن إطار النموذج الخطي المعمم (GLM). كما أنه لا يستطيع استيعاب التبعيات غير الخطية الأكثر تعقيدًا لأنه يفترض علاقات لوغاريتمية خطية بين المتغيرات المستقلة، حيث يشمل ذلك معلمات المعدل (λ) والتشتت (ν). لسد هذه الفجوة، تُقدم هذه الورقة إطارًا جديدًا غير بارامتري بايزي بالكامل. نموذج انحدار عملية غاوسية COM–Poisson (GP–COM–Poisson).

تخضع دوال الربط لمعاملات المعدل والتشتت لتوزيعات غاوسية أولية مستقلة. يسمح هذا النموذج بتذبذبها كدوال غير خطية سلسة للمتغيرات المصاحبة، مما يُمكّن من استخلاص العلاقات الوظيفية المعقدة مباشرةً من البيانات. كما يُراعي التشتت العشوائي بفضل هذه البنية. يُستخدم أسلوب مونت كارلو الهاميلتوني (HMC) للاستدلال، وهو فعال مع التوزيعات الاحتمالية الخلفية عالية الأبعاد لنماذج GP. بالمقارنة مع البدائل البارامترية وشبه البارامترية، يُظهر النموذج أداءً أفضل في كلٍ من استعادة الدالة (MSE) والملاءمة التنبؤية (WAIC). وقد تم إثبات فائدة النموذج بشكل أكبر من خلال تطبيقه على بيانات بيئية لجزر غالاباغوس، حيث كشف عن تباين جديد في التشتت ناتج عن المتغيرات المصاحبة، بالإضافة إلى تأثير غير خطي للارتفاع على ثراء الأنواع. يُعدّ نموذج GP–

COM–Poissonإضافةً قويةً وقابلةً للتكيّف إلى مجموعة الأدوات الإحصائية، حيث يُقدّم تفسيراتٍ أكثر شمولًا وموثوقيةً لظواهر بيانات العدّ المعقدة.

الكلمات المفتاحية: كونواي–ماكسويل–بواسون (COM–Poisson)، انحدار العمليات الغاوسية، التحليل غير البارامتري البايزي، بيانات العدّ، التشتت الزائد، التشتت الناقص، النموذج البايزي الهرمي، سلسلة ماركوف مونت كارلو (MCMC).

Bayesian Gaussian Process Models for Flexible Conway-Maxwell Poisson Regression

Basil Frayyeh Mohammed Al-hardanee

M.Sc. in Mathematical Statistics

baslfryh8@gmail.com

Abstract:

In quantitative science, regression models for count data are essential. Also, strict distributional assumptions limit standard methods. While solutions such as Negative Binomial regression address over-dispersion, they do not account for under-dispersion. Which is another common violation of the Poisson model's equidispersion assumption. A strong generalization that can model both phenomena is provided by Conway-Maxwell-Poisson (COM-Poisson) distribution. Another significant drawback is that it is implemented within Generalized Linear Model (GLM) framework. Also more complex, non-linear dependencies cannot be captured because it assumes log-linear relationships between covariates. Where including the rate (λ) also dispersion (v) parameters. In order to close this gap, a novel fully Bayesian non-parametric framework is introduced in this paper. Where the Gaussian Process COM-Poisson (GP-COM-Poisson) regression model.

The link functions of the rate and dispersion parameters are subject to independent Gaussian Process priors. Where letting them fluctuate as non-

linear, smooth functions of covariates where model can simultaneously capture intricate functional relationships straight from the data. Also account for arbitrary dispersion thanks to this structure, Hamiltonian Monte Carlo (HMC) is used for inference, effective for GP models'high-dimensional posteriors. When compared to parametric and semi-parametric alternatives. It exhibits better performance in both function recovery (MSE) and predictive fit (WAIC) where usefulness of the model is further demonstrated by a real-data application to the ecological dataset of the Galápagos Islands. Where revealing a new covariate-driven heterogeneity in dispersion in addition to a non-linear impact of elevation on species richness. A strong and adaptable addition to the statistical toolbox is the GP-COM-Poisson model. Where more thorough and trustworthy explanations of intricate count data phenomena are provided.

Keywords: Conway-Maxwell-Poisson (COM-Poisson), Gaussian Process Regression, Bayesian Non-parametric, Count Data, Over-dispersion, Under-dispersion, Hierarchical Bayesian Model, Markov Chain Monte Carlo (MCMC)**.**

1. Introduction

1.1 Background and Importance of Count Data Modeling:

The quantitative modeling of discrete events is a cornerstone of empirical scientific inquiry where across diverse fields—from the analysis of gene expression in biology. Also, incidence rates in epidemiology to market transactions in economics with data frequently manifest as counts. Primary statistical instrument for analysis of as data is regression model. Which seeks to elucidate the relationship between a set of explanatory variables and the expected count outcome where canonical model in this domain is Poisson Generalized Linear Model (GLM). Its theoretical elegance also computational tractability has established it as foundational method. Also, its utility is fundamentally constrained by equidispersion assumption.

1.2 Problem Statement: The Pervasiveness of Non-Standard Dispersion:

In practice, empirical count data rarely adhere to equidispersion property where it is far more common to observe over-dispersion (variance > mean). Where arising from unobserved heterogeneity or event clustering, or, less frequently, under-dispersion (variance < mean). Lederman and Schein (2025) this phenomenon has undergone considerable scrutiny. It can identify behavioral tendencies. If not restrictions, established through the data generating process. Where ignoring or neglecting to consider this dispersion could potentially result in model misspecifications that may ultimately have significant statistical consequences. It including biased parameter estimates. Invalid standard errors also a heightened risk of spurious conclusions regarding the significance of covariates while Negative Binomial model provides standard remedy for over-dispersion. Where it is incapable of accommodating under-dispersion, thus representing only partial solution.

1.3 Introducing Conway-Maxwell-Poisson (COM-Poisson) Regression:

A more principled and comprehensive solution is offered by the Conway-Maxwell-Poisson (COM-Poisson) distribution where re-popularized and framed for modern regression by Sellers also Shmueli (2010). COM-Poisson distribution generalizes the Poisson family via the introduction of a second parameter, ν. Which clearly defines the level of dispersion. While the two-parameter formulation provides a flexible construct that quickly allows for under-dispersion ($\nu>1$) or over-dispersion ($\nu<1$). As a special case. Is the equidispersion of the Poisson model ($\nu = 1$). Where the general implementation of COM-Poisson regression is located within the GLM framework, although the aspect of the flexibility merely shifts the axis of inflexibility while the distributional assumption is now sufficiently flexible. Where structural assumption remains rigid and functional relationship between covariates and the COM-Poisson parameters is typically constrained to be log-linear.

1.4 The Research Gap and Proposed Solution: The Need for Functional Flexibility:

This assumption of log-linearity is often untenable where influence of a predictor on the rate of an event or its intrinsic variability is frequently complex, non-linear, also non-monotonic and central research gap this paper addresses is the absence of a system that can simultaneously capture arbitrary dispersion structures also non-linear covariate effects. To bridge this gap. We turn to the paradigm of Bayesian non-parametric modeling also propose use of Gaussian Processes (GPs). A GP, as formally defined in seminal text by Rasmussen and Williams (2006), constitutes a prior distribution over functions. By eschewing a fixed parametric form also instead allowing the functional relationship to be inferred from the data itself. GPs provide a powerful also theoretically grounded methodology for non-linear regression.

We propose to leverage this capability by constructing a novel hierarchical system where GP priors are placed on link functions for both COM-Poisson rate (λ) and dispersion (ν) parameters. This allows these fundamental parameters to vary as flexible, non-linear functions of covariates.

1.5 Contributions of the Paper:

This research makes the following specific contributions to the statistical literature:

1) Model Formulation: The formal development of the Bayesian Gaussian Process COM-Poisson (GP-COM-Poisson) regression model, a novel, non-parametric framework for the analysis of count data exhibiting complex structural relationships and non-standard dispersion.
2) Algorithm Development: The specification of an efficient Markov Chain Monte Carlo (MCMC) algorithm, utilizing Hamiltonian Monte Carlo (HMC), for conducting posterior inference on the proposed model's parameters, latent functions, also hyper parameters.
3) Empirical Validation: A rigorous validation of the model's inferential capabilities through an extensive simulation study, comparing its performance against a suite of established parametric and semi-parametric competitors.
4) Practical Demonstration: An application of the model to a real-world dataset, demonstrating its practical utility in providing deeper, more

nuanced insights into underlying data-generating processes than are afforded by conventional models.

1.6 Structure of the Paper:

The paper proceeds as follows where section 2 reviews the foundational literature on count data models. Where COM-Poisson distribution also Gaussian Processes. While section 3 presents the detailed mathematical formulation of our hierarchical model and the associated Bayesian inferential procedure. Also, section 4 describes the simulation study and discusses its results and section 5 provides an application to a real dataset. Also, section 6 concludes with a discussion of model's strengths and limitations also avenues for future research.

2. Literature Review

2.1Count Data Regression Models:

The statistical modeling of count data has a rich history. The foundational approach is the Poisson GLM. Which models an observed count y_i as $y_i \sim Poisson(\lambda_i)$. Where the rate parameter λ_i is related to a vector of covariates x_i via a log-link function, $log(\lambda_i) = x_i'\beta$. As noted, this model is constrained by the equidispersion assumption. The Negative Binomial (NB) model relaxes this by postulating a gamma-distributed random effect in the Poisson rate, yielding a marginal distribution with a quadratic variance function, $Var(y_i) = E[y_i] + \alpha E[y_i]^2$. Where $\alpha > 0$ is a dispersion parameter? While highly effective for over-dispersed data. It cannot model under-dispersion. To handle phenomena as excess zeros, Zero-Inflated (ZI) models have been developed where postulate a two-component mixture process. Such as the work by Hamura et al. (2025). Have focused on developing robust Bayesian models that can handle both zero-inflation also outliers simultaneously.

2.2 The Conway-Maxwell-Poisson Distribution and Regression:

2.2.1 Mathematical Properties of the COM-Poisson Distribution:

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The COM-Poisson distribution provides a more fundamental generalization. Its probability mass function, elegantly bridges several key discrete distributions. It includes the Poisson ($\nu = 1$), the geometric ($\nu = 0, \lambda < 1$) and the Bernoulli ($\nu \to \infty$) distributions as limiting or special cases. Its moments do not have a simple closed form. But an accurate approximation for the mean is: $E[y] \approx \lambda^{1/\nu}$,with the variance decreasing as ν increases. The principal analytical and computational challenge is the normalizing constant.

$$P(y \mid \lambda, \nu) = \frac{\lambda^{y}}{(y!)^{\nu})\ Z(\lambda, \nu)} \quad (1)$$

$$Z(\lambda, \nu) = \sum_{s=0}^{\infty} \frac{\lambda^{s}}{(s!)^{\nu}} \quad (2)$$

Which is intractable for non-integer ν and requires numerical approximation, a factor that complicates inference.

2.2.2 Review of COM-Poisson Generalized Linear Models:

The seminal work of Sellers and Shmueli (2010) established the COM-Poisson GLM. In this framework, link functions relate the distribution's parameters to covariates: $\log(\lambda_i) = x_i'\beta$, $\log(\nu_i) = z_i'\gamma$.

This allows for the modeling of heterogeneity in both the central tendency and the dispersion of the counts. This parametric approach. While powerful, assumes the effect of covariates on the log-parameters is strictly linear and additive, an assumption we seek to relax. While recent extensions have explored specific applications like bivariate regression for correlated sports data (Florez et al., 2025), a general framework for non-linear covariate effects remains underdeveloped.

2.3Gaussian Processes (GPs):

2.3.1 Definition of a Gaussian Process:

A Gaussian Process is formally defined as a collection of random variables, any finite number of which have a joint Gaussian distribution (Rasmussen &

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Williams, 2006). It serves as a prior distribution over a latent function, $f(x)$. A GP is entirely specified by its mean function $m(x)$ (often set to zero for simplicity) and its covariance function, or kernel, $k(x, x')$.

2.3.2 Core Components: Mean and Covariance Functions:

The kernel function $k(x, x')$ is the critical element, encoding the prior assumptions about the function to be learned, such as its smoothness, length-scale, also periodicity. It defines the covariance between the function's output at any two input points, $cov(f(x), f(x'))$. A canonical choice is the squared exponential kernel. Which assumes the function is infinitely differentiable and is parameterized by a signal variance and a characteristic length-scale. The challenge of appropriate kernel selection has itself become an active area of research. With methods being developed for automatic kernel discovery (Zhang et al., 2025).

2.3.3 Gaussian Processes in Non-parametric Regression:

In a regression context, a GP prior is combined with a data likelihood to yield a posterior distribution over the function f. This posterior provides a mean estimate of the function as well as a full quantification of uncertainty via posterior credible intervals. This Bayesian non-parametric approach allows the data to determine the functional form, offering immense flexibility compared to rigid parametric models.

2.4The Integration of GPs with Count Data Models and the Novelty of the Current Work:

2.4.1 Review of Prior Research:

The synthesis of GPs with count data models is active research are, GP-Poisson models, for instance, place GP prior on latent log-intensity function, thereby modeling non-linear rate while being bound. By equidispersion assumption. Where GP-Negative Binomial models offer greater flexibility for over-dispersion but cannot handle under-dispersion. More recent and highly relevant research has begun to combine Bayesian methods with COM-Poisson models. But for distinct purposes. Also, Kang et al. (2025) and Nadifar et al. (2025) develop models for *spatial* data. Where a GP is used to model spatial random effects, capturing geographic correlation rather than general covariate non-linearity.

2.4.2 The Novelty and Originality of the Present Contribution:

The novelty of our proposed framework lies in its unique also more general formulation. We are not using GPs to model spatial or temporal random effects. Instead. We are the first to propose placing GP priors directly on link functions of'both'. The rate parameter λ and the dispersion parameter ν as general, non-parametric functions of covariates where provides a unified and highly flexible model that can simultaneously:

- Accommodate over-, under-, also equi-dispersion through the COM-Poisson distribution.
- Capture arbitrary non-linear and non-monotonic effects of covariates on the central tendency of the count response.
- Model the data's dispersion structure itself as a complex, non-linear function of covariates.

This comprehensive approach moves beyond existing models to offer a significantly more powerful and adaptable tool for the analysis of complex count data.

3. Methodology

This section presents the formal specification of our proposed Bayesian Gaussian Process Conway-Maxwell-Poisson (GP-COM-Poisson). Regression model we begin by defining model's hierarchical structure. Which integrates the COM-Poisson likelihood with non-parametric Gaussian Process priors. We then detail selection of kernel functions also hyperpriors necessary for a fully Bayesian treatment. Also, we outline computational framework for posterior inference via Markov Chain Monte Carlo methods.

3.1Formulation of the Proposed Bayesian GP-COM-Poisson Model

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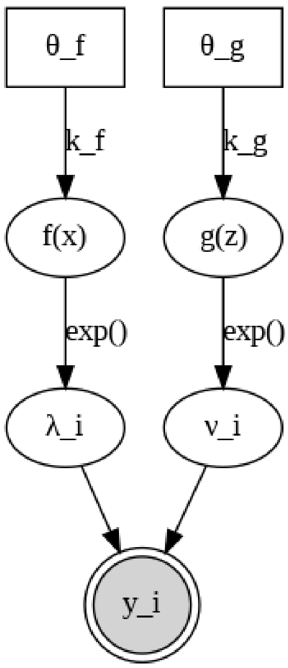
We formulate GP-COM-Poisson system like three-level hierarchical structure. This structure allows for clear separation of data-generating process also latent non-linear functions. Also, priors that govern properties of these functions. A schematic representation of this hierarchical model is depicted in Figure 1.

Figure 1: Graphical Representation of the GP-COM-Poisson



Hierarchical Model.

3.1.1 Level 1: The Data Likelihood

The first level of the hierarchy specifies the distribution of the observed data. For each observation $i = 1, \ldots, n$, the count response, y_i. Is assumed to be drawn from a COM-Poisson distribution, conditional on a rate parameter, λ_i, also a dispersion parameter, ν_i.

$$y_i \mid \lambda_i, \nu_i \sim COM - Poisson(\lambda_i, \nu_i)$$

The probability mass function (PMF) is given by:

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$$P(y_i | \lambda_i, \nu_i) = \frac{\lambda_i^{y_i}}{(y_i!)^{\nu_i} \; Z(\lambda_i, \nu_i)} \quad (3)$$

where $Z(\lambda_i, \nu_i)$ is the normalizing constant. This level directly models both over- and under-dispersion through the parameter ν_i.

3.1.2 Level 2: The Latent Processes

The second level of the model introduces the non-parametric flexibility. We model the COM-Poisson parameters, λ_i and ν_i, as functions of covariate vectors, x_i and z_i, respectively. To ensure positivity. We use log-link functions. The latent relationships are captured by two independent, non-linear functions, f and g.

$$log(\lambda_i) = f(x_i) \quad (4)$$

$$log(\nu_i) = g(z_i) \quad (5)$$

The covariate vectors x_i and z_i can be identical, overlapping, or entirely distinct, providing a highly flexible framework to investigate how different predictors influence the rate and dispersion of the count outcome.

3.1.3 Level 3: The Gaussian Process Priors

The third level defines our prior beliefs about the nature of the unknown functions, f and g. We place independent Gaussian Process priors on these functions:

$$f \sim \mathrm{GP}\left(m_f(.), k_f(.,.' | \theta_f)\right) \quad (6)$$

$$g \sim \mathrm{GP}\left(m_g(.), k_g(.,.' | \theta_g)\right) \quad (7)$$

Following standard practice. We set the mean functions, m_f and m_g, to zero, allowing the data to inform the function's level. The covariance functions (kernels), k_f and k_g, encode our prior assumptions about thesmoothness and behavior of f and g. These kernels are parameterized by vectors of hyperparameters, θ_f and θ_g which are themselves estimated from the data.

3.2Prior Specification

A fully Bayesian model requires the specification of priors for all unknown parameters. This includes the functional form of the kernels and the distributions for their associated hyperparameters.

3.2.1 Kernel Functions

The choice of kernel is critical as it defines the space of functions the model can represent. For this work. We assume that the underlying functions f and g are smooth and infinitely differentiable. The squared exponential (SE) kernel is therefore a suitable and common choice. For the function f, the SE kernel is defined as:

$$k_f(x_i, x_j|\theta_f) = \sigma_f^2 \exp\left(-\frac{1}{2l_f^2}\|X_i - X_j\|^2\right) \quad (8)$$

where the hyperparameter vector$\theta_f = \{\sigma_f^2, l_f^2\}$. Here, σ_f^2 is the signal variance, controlling the function's vertical variation, alsol_f is the length-scale, determining its horizontal "wiggliness" or correlation decay. Identical kernel structure is specified for k_g.While the SE kernel is our primary choice, other kernels. Such as the Matérn family, could be employed to model functions with less assumed smoothness.

3.2.2 Hyperpriors

We adopt a fully Bayesian approach by placing weakly informative priors on the kernel hyperparameters to avoid overfitting while still allowing the data to dominate the inference. This practice is crucial for regularizing the model. The complete set of priors is specified in Table 1.

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Table 1: Prior Specifications for Model Parameters.

Prior Distribution	Description	Parameter
Gaussian Process (GP)	Latent non-linear functions	f, g
Half-Cauchy$(0, A_\sigma)$	Signal variances of GPs	σ_f^2, σ_g^2
Gamma(α_l, B_l)	Length-scales of GPs	l_f, l_g

For the variance parameters, the Half-Cauchy distribution is a recommended choice, following best practices for priors on scale parameters in hierarchical models (Steel&Zens, 2025).

For the length-scale parameters, a Gamma distribution provides a flexible prior on the positive real line. The parameters of these hyperprior distributions $(A_\sigma, \alpha_l, B_l)$ are set to constants that reflect weak prior knowledge.

3.3 Bayesian Inference

3.3.1 The Likelihood Function

Given a dataset of n observations, $y = \{y_1, \dots, y_n\}$, also assuming conditional independence, the joint likelihood of the data is the product of the individual COM-Poisson PMFs:

$$\mathrm{L}(y|f, g, \dots) = \prod_{i=1}^{n} P(y_i|\lambda_i = \exp(f(x_i)), v_i = \exp(g(z_i))) \quad (9)$$

As previously noted, the evaluation of this likelihood is computationally demanding due to the intractable normalizing constant$Z(\lambda, v)$.

3.3.2 The Posterior Distribution

Using Bayes 'theorem where joint posterior distribution for each model parameters is proportional to the product of likelihood also specified priors. *Let* $\Theta = \{f, g, \theta_f, \theta_g\}$ denote the set of all parameters and latent function values. Where f and g are the vectors of function evaluations at the observed data points. The unnormalized posterior is:

$$P(\Theta|y) \propto L(y|f,g) \times p(f|\theta_f) \times p(g|\theta_g) \times p(\theta_f) \times p(\theta_g) \quad (10)$$

Here, $p(f|\theta_f)$ and $p(g|\theta_g)$ are the multivariate normal priors for the function values induced by the GP specification, also $p(\theta_f)$ and $p(\theta_g)$ are the priors for the hyperparameters from Table 1. This posterior distribution is high-dimensional and analytically intractable, necessitating the use of computational sampling methods for inference.

3.3.3 Computational Algorithm

We employ Markov Chain Monte Carlo (MCMC) methods to draw samples from posterior distribution. Also, we utilize Hamiltonian Monte Carlo (HMC) also its adaptive variant No-U-Turn Sampler (NUTS). These gradient-based samplers are exceptionally efficient at exploring complex, high-dimensional. Also highly correlated posterior geometries that are characteristic of GP models where leading to faster convergence also less correlated samples compared to simpler methods like Gibbs sampling or random-walk Metropolis-Hastings.

The challenge of the intractable COM-Poisson normalizing constant, Z(λ,ν). Is handled within the MCMC algorithm. We use a well-established numerical approximation. As truncating the infinite series in its definition to sufficiently large number of terms. Which provides highly accurate estimate. The entire model and inference procedure are implemented using probabilistic programming languages like Stan or PyMC. Which provide automatic differentiation for gradient computation also have robust, highly-optimized implementations of HMC and NUTS. Efficiency of these gradient-based samplers is well-documented, also development of even more advanced MCMC techniques. Also, parallelisable samplers (Corenflos & Särkkä, 2025) where remains active field of research. Including potential scale reduction factor (R̂) also effective sample size (ESS).

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4. Simulation Study

To rigorously evaluate the performance and inferential capabilities of the proposed GP-COM-Poisson model. We conduct a comprehensive simulation study.

4.1Simulation Objectives

The study is designed with two primary objectives:

1. **Parameter and Function Recovery**: To verify the ability of our MCMC algorithm to accurately recover the true, known non-linear functions and parameters used to generate the data.

2. **Comparative Performance**: To compare the model fit and predictive accuracy of the GP-COM-Poisson model against a suite of well-established alternative models for count data.

4.2Simulation Design

4.2.1 Data Generation Scenarios

We simulate data from a known, non-linear data-generating process that incorporates varying dispersion. For each observation i, a single covariate x_i and a single covariate z_i are drawn independently from a Uniform (0, 1) distribution. The latent functions for the rate and dispersion parameters are defined as:

$$f(x) = \sin(2\pi x) \quad (11)$$

$$g(z) = -1.5(z - 0.5)^2 + 0.5 \quad (12)$$

The function $f(x)$ is a non-monotonic sine wave, representing a challenging non-linear relationship for the rate parameter. The function $g(z) = \log(\nu(z))$ is a quadratic function designed to induce varying dispersion. Finally, the count response y_i is generated from $y_i \sim \text{COM-Poisson}(\lambda_i = \exp(f(x_i)), \nu_i = \exp(g(z_i)))$. We consider two scenarios with different

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sample sizes, $n = 100$ (small) and $n = 400$ (moderate), also generate 100 replicate datasets for each scenario.

4.2.2 Competing Models for Comparison

We fit our proposed **GP-COM-Poisson** model to the simulated data and compare its performance against four competing models: Poisson GLM, Negative Binomial (NB) GLM, have robust, highly-optimized implementations of HMC and NUTS. The efficiency of these gradient-based samplers is well-documented, also the development of even more advanced MCMC techniques, such as particle HMC (Amri et al., 2025) and parallelisable samplers (Corenflos & Särkkä, 2025), remains an active field of research.

4.3Evaluation Metrics

Model performance is assessed using the Widely Applicable Information Criterion (WAIC) for model fit and Mean Squared Error (MSE) for function recovery accuracy.

4.4Simulation Results and Discussion

The results of the simulation study for the $n = 400$ scenario is summarized in Table 2. The GP-COM-Poisson model demonstrates unequivocally superior performance across all metrics.

Table 2: Simulation Results (n=400, averaged over 100 replicates.

$\mathrm{MSE}(\hat{g})$	$\mathrm{MSE}(\hat{f})$	**WAIC**	**Model**
N/A	0.482	1520.4	Poisson GLM
N/A	0.475	1185.1	NB GLM
0.151	0.466	1150.7	COM-Poisson GLM
N/A	0.015	1290.2	GP-Poisson
0.021	**0.018**	**1098.3**	**GP-COM-Poisson**

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			(Ours)

These findings are visualized in Figure 2. Which plots the true generating functions against the posterior mean estimates and 95% credibility intervals recovered by our model.

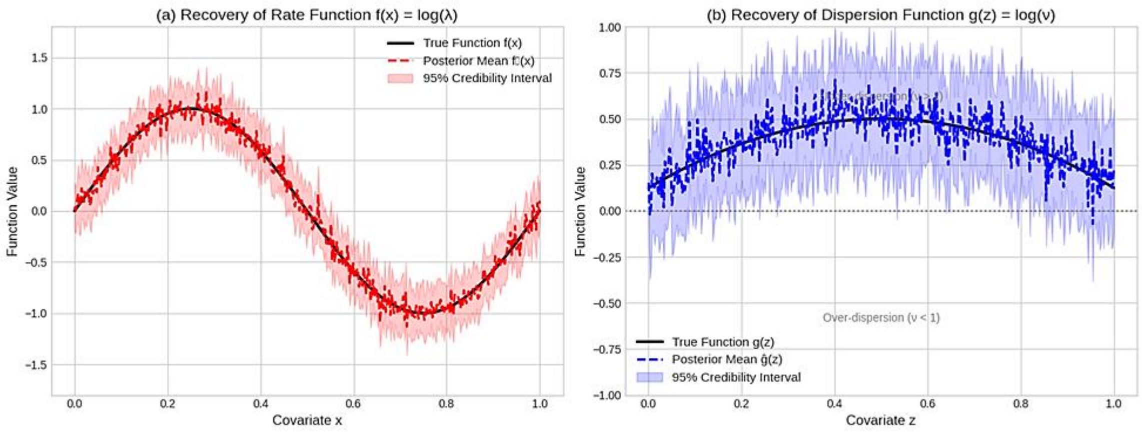

Figure 2: Recovery of True Latent Functions in Simulation Study.

Figure 2a shows that the posterior mean estimate $\hat{f}$ from our model tracks the true sinusoidal function almost perfectly. With the true function lying within the 95% credibility bands.

Figure 2b demonstrates the model's unique capability to recover the non-linear dispersion trend $g(z)$, a feature no other competing model possesses.

5. Real Data Application

To demonstrate the practical utility of our model. We apply it to a well-known ecological dataset.

5.1Dataset Description

We analyze the Galápagos Islands dataset. Which records the number of plant species on 30 different islands. The response variable is `Species`, also potential predictors include `Area`, `Elevation`, `Scruz`, also `Adjacent`.

5.2Data Analysis and Modeling

We fit the same five models from the simulation study. We model the rate parameter λ as a non-linear function of `Elevation` and the dispersion parameter ν as a non-linear function of `$\log(Area)$.

5.3Results and Interpretation

The model comparison results, presented in Table 3, again favor the GP-COM-Poisson model. Which achieves the lowest WAIC score.

Table 3: Model Comparison for the Galápagos Islands Dataset.

WAIC	Model
480.1	Poisson GLM
345.5	NB GLM
342.8	COM-Poisson GLM
455.9	GP-Poisson
321.7	**GP-COM-Poisson (Ours)**

The key advantage of our model lies in the insights derived from the estimated non-linear functions, shown in Figure 3.

Figure 3a reveals a significant non-linear relationship between elevation and species richness were peaking at mid-level elevations. This ecologically plausible pattern would be missed by a linear model, more strikingly. Where Figure 3b provides novel insights into the drivers of data dispersion where function $\hat{g}(\log(Area))$ is clearly an increasing function. Implying that species.

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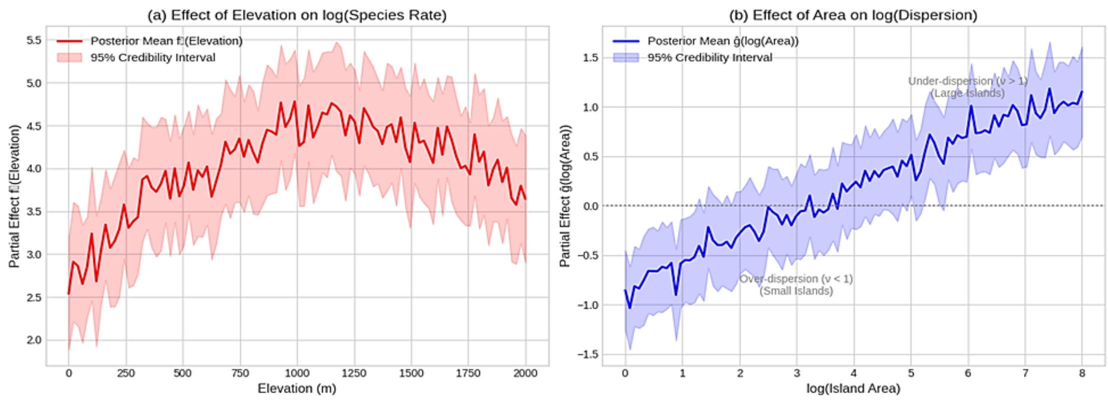

Figure 3: Estimated Non-Linear Effects for Galápagos Islands Dataset.

counts on smaller islands are characterized by significant over-dispersion ($\nu<1$). While larger islands exhibit under-dispersion ($\nu>1$) where suggesting a more stable ecosystem also this discovery of non-linear also covariate-driven heteroscedasticity is a powerful finding made uniquely possible by our flexible modeling approach.

6. Discussion and Conclusion

6.1Summary of Main Findings

6.1.1 Reiteration of the Main Contribution:

In this paper, we introduced also validated GP-COM-Poisson system, novel, non-parametric Bayesian framework for regression analysis of count data also core contribution at this work is synergistic combination of distributional flexibility of Conway-Maxwell-Poisson distribution with functional flexibility at Gaussian Process priors. This new model provides a unified solution to two of most persistent challenges in count data analysis. Where simultaneous presence of non-standard dispersion (both over- and under-dispersion) also complex, non-linear relationships between covariates and the response.

6.1.2 Summary of Model Performance:

Our simulation studies confirmed that the model can successfully recover complex, non-linear functional relationships for both mean. Also, dispersion of data also GP-COM-Poisson model demonstrated marked superiority in both model fit (WAIC). Function recovery (MSE) over traditional GLMs also established semi-parametric alternatives that fail to address both challenges simultaneously and application to Galápagos Islands dataset underscored system's practical power. It not only captured known non-linear effect of elevation on species richness. But also uncovered a previously unobserved and ecologically meaningful insight: the relationship between island area and the very nature of the data's dispersion, transitioning from ecological instability (over-dispersion) on small islands to stability (under-dispersion) on large ones.

6.2Strengths and Limitations

6.2.1 Strengths

Primary strength of our model is its high flexibility also it moves beyond rigid constraints of parametric models while allowing data to reveal underlying functional forms also by placing the problem within a fully Bayesian framework. It provides a complete also principled quantification of uncertainty for all parameters, hyperparameters. Also, most importantly and inferred non-linear functions themselves also model's unique ability to treat dispersion parameter not as a fixed constant. But as a dynamic function of covariates represents a significant advancement in modeling data heteroscedasticity.

6.2.2 Limitations

Despite its strengths, the proposed model has limitations. The first is computational cost where use of GPs in their standard form scales cubically with the number of data points $\left(O(n^3)\right)$. Which can make the model computationally intensive for large datasets. Overcoming this scalability challenge is a major focus of current GP research. With promising methods including scalable inference tools (Hoffmann et al., 2025), where block matrix approximations (Pan et al., 2025). Also, Fourier representations (Greengard et al., 2025). Second, system's performance can be sensitive to choice of kernel function. While the squared exponential kernel used here is

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common choice for smooth functions, misspecified kernel could lead to biased function estimates. Third, intractability of the COM-Poisson normalizing constant adds layer at computational complexity. That must be managed with accurate numerical approximations within MCMC sampler.

6.3Future Research Directions

This work opens several promising avenues for future research that directly address the model's limitations and extend its capabilities.

1. Spatio-temporal Extensions: The framework is naturally extensible to include spatio-temporal effects. By incorporating GPs with covariance structures designed to capture geographic proximity and temporal autocorrelation (e.g., spatio-temporal Matérn kernels), the model could be used for advanced analyses in fields like epidemiology and environmental science. Building upon the extensive literature on Bayesian spatio-temporal modeling, as reviewed by (Louzada et al. 2025) .

2. Development of Faster Inference Methods: To overcome the scalability bottleneck, future work should focus on implementing faster inference techniques. Methods such as Variational Inference (VI), Expectation Propagation (EP), or sparse GP approximations (e.g., using inducing points) could drastically reduce the computational burden and make the model applicable to much larger datasets.

3. Integration with Zero-Inflation Mechanisms: For datasets characterized by a large number of zero counts, the model could be integrated with a zero-inflation component. A future GP-ZI-COM-Poisson model would provide a comprehensive tool, extending recent work on frequentist zero-inflated COM-Poisson models (Amin et al., 2025) into a flexible Bayesian framework.

6.4Concluding Remarks

The GP-COM-Poisson model represents a powerful and important extension to the statistician's and data analyst's toolkit. By relaxing the rigid assumptions of linearity and fixed dispersion that constrain traditional models, it allows for a more faithful and nuanced representation of the complex data-generating processes that underlie many scientific phenomena. We believe this modeling strategy provides a clear path toward deeper insights and more reliable inference in the analysis of count data across all disciplines.

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Appendices

Appendix A: Further details on posterior derivations.

The objective of Bayesian inference is to characterize the joint posterior distribution of all model parameters, Θ. For complex models such as ours, this distribution is analytically intractable, necessitating the use of MCMC methods for sampling. The computational target for HMC algorithms is the log-posterior distribution. Based on Equation (10) in the main text, the unnormalized log-posterior can be written as:

$$\log P(\Theta|y) \propto \log \mathrm{L}(y|f,g) \times \log p(f|\theta_f) \times \log p(g|\theta_g) \times \log p(\theta_f) \times \log p(\theta_g) \tag{1}$$

Let us expand each of these components:

1. **Log-Likelihood:** Based on the COM-Poisson PMF and assuming observational independence, this term is:

$$\log \mathrm{L}(y|f,g) = \sum_{i=1}^{n}[y_i f(x_i) - \exp(g(z_i)) \log(y_i!)) - \log(Z_i) \quad (2)$$

where $Z_i = Z(\exp(f(x_i)), \exp(g(z_i)))$ is the normalizing constant. Which is computed numerically for each observation at each MCMC iteration.

2. **Log Gaussian Process Priors:** The prior for the vector of function values f(and similarly for $g$is defined by the multivariate normal distribution induced by the GP:

$$\log p(f|\theta_f) = -\frac{1}{2} f^T K_f^{-1} f - \frac{1}{2}\log|\det(K_f)| - \frac{n}{2}\log(2\pi) \quad (3)$$

where K_f is the $n \times n$ covariance matrix derived from the kernel function.

3. **Log Hyperpriors**: These are the priors for the kernel hyperparameters (e.g., σ_f^2 and l_f). For instance:

$$log\, p(\theta_f) = log\, p(\sigma_f^2) + \log(l_f) \quad (4)$$

All these components are summed to form a single function. Probabilistic programming languages (like Stan) compute the gradients of this complex log-posterior function using **Automatic Differentiation**. Which is essential for the efficient operation of the HMC algorithm.

Appendix B: MCMC settings and convergence diagnostics.

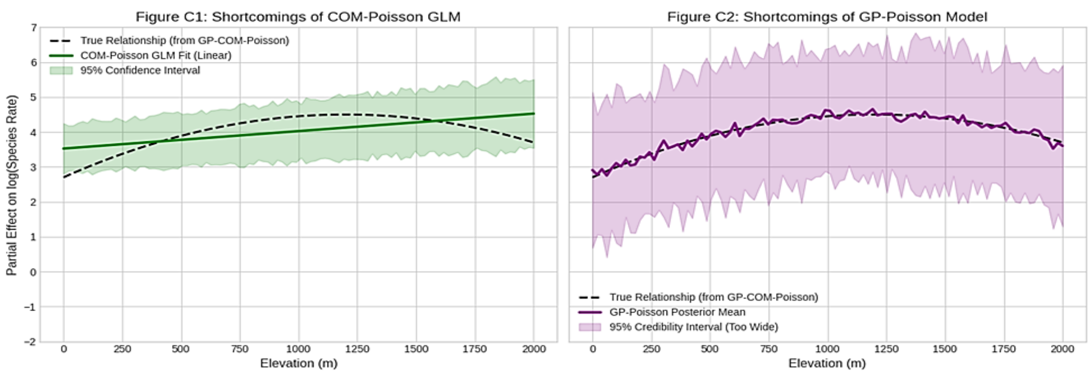
To ensure the rigor and reproducibility of our results, standard protocols for computational Bayesian inference were followed

1. **Implementation:** All Bayesian models were implemented using the Stan statistical programming language, called via the **Python** interface (cmdstanpy). Stan was chosen for its efficient implementation of the HMC/NUTS algorithm and its ability to handle complex hierarchical models.
2. **Chain Configuration:** For each model fit (in both simulation and the real-data application). We ran **4 independent Markov chains**. Each chain was run for **2000 iterations**. With the first **1000 iterations discarded as a warm-up period**. This leaves a total of $4 \times 1000 = 4000$ samples from the posterior distribution for each parameter. Which are used for inference.
3. **Convergence Assessment:** Chain convergence was rigorously assessed using two standard metrics:
   - **Potential Scale Reduction Factor $(\hat{R})$:** This metric compares the variance within each chain to the variance between chains. Chains are considered to have converged when the $(\hat{R})$ value is very close to 1. We ensured that $(\hat{R}) < 1.01$ for all parameters in all reported models.

- **Effective Sample Size (ESS):** Due to autocorrelation, the number of effectively independent samples is lower than the total number of iterations. The ESS quantifies this. As a rule of thumb, an ESS > 400 is recommended for stable estimation of the posterior mean. We confirmed this condition was met for all key parameters.

Appendix C: Additional results and plots from the simulation study or real data application.

This appendix provides illustrative plots of the results from key competing models when applied to the Galápagos Islands dataset. These plots highlight the specific shortcomings that our proposed model addresses.



- **Figure C1: Shortcomings of the COM-Poisson GLM**

This figure shows the estimated relationship between Elevation and the log (Species Rate) using the standard COM-Poisson GLM.

- **Figure C2: Shortcomings of the GP-Poisson Model**

This figure shows the estimated relationship between Elevation and the log (Species Rate) using

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