



(677) - (699)

العدد الرابع
والأربعون

متباينات تكامل سوغينو للدوال المحدبة:
حدود جديدة من نوع ساندور وتطبيقاتها
ولاء محمود عباس
feras1997feras1997@gmail.com

المستخلص:

يتناول هذا البحث إمكانية توسيع متباينات التحدب الكلاسيكية إلى القياسات غير الجمعية، وذلك في ضوء التطورات الحديثة والحاجة المتزايدة إلى حدود تكاملية دقيقة في نماذج التجميع. تطوّر هذه الدراسة حدوداً جديدة (سفلى وعلياً) لتكامل سوغينو الضبابي المعروف على فترات حقيقية مغلقة من الدوال المحدبة، والمحدبة من النوع S، وشبه المحدبة. وتُمثّل هذه النتائج تنقيحاً مشروعاً للتنقيح الذي أثبتته ساندور عام ١٩٩٧ لتكامل ريمان الاعتيادي. تعتمد البراهين على تحليل المجموعات المستوية، الذي أثبت فعاليتها العالية في إبراز البنية الأصغرية - الأعظمية لتكامل سوغينو، ولم يُستخدم أي من البراهين التقاربية الرتيبة الكلاسيكية في اشتقاق هذه النتائج. وإلى جانب البناء النظري، تُقدّم سلسلة من الأمثلة العددية المحولة وجدول مقارنة مختصر يتيحان للقارئ تقييم الحجم النسبي للمناطق الحديثة لعدد من عائلات الدوال النموذجية. كما نناقش العلاقة بين متباينات هيرميت - هادامار وتكاملات شوكيه ونتائجنا، ونختتم البحث بطرح جملة من المسائل التي نرى أنها تستحق مزيداً من البحث مستقبلاً.

الكلمات المفتاحية: تكامل سوغينو، القياس الضبابي، الدالة المحدبة، متباينة من نوع ساندور، متباينة هيرميت - هادامار، التحدب من النوع S، طريقة المجموعات المستوية، القياس الرتيب.

**Sugeno Integral Inequalities for Convex Functions:
New Sandor-Type Bounds and Their Ramifications****Walaa Mahmood Abbas****feras1997feras1997@gmail.com****Abstract**

Motivated by recent developments and the growing demand for precise integral bounds in aggregation models, this work investigates the extension of classical convexity inequalities to non-additive measures. We establish new lower and upper bounds for the Sugeno integral of convex, s -convex, and quasi-convex functions defined on compact real intervals. These bounds constitute genuine analogues, within the non-additive setting, of the refinement established by Sandor in 1997 for the classical Riemann integral. The proofs rely on level-set analysis, which proves particularly effective in capturing the minimum–maximum structure of the Sugeno integral; none of the results invoke the classical monotone-convergence arguments. Beyond the theoretical development, we provide a series of worked numerical examples and a compact comparison table, enabling the reader to assess the relative size of the bounding regions for several representative families of functions. We further discuss the connections between our results and both the Hermite–Hadamard inequality and the Choquet integral, and we conclude by posing several problems that merit future investigation.

Keywords: Sugeno integral · fuzzy measure · convex function · Sandor-type inequality · Hermite–Hadamard inequality · s -convexity · level-set method · monotone measure.

1. Introduction and Motivation

When Sugeno defended his doctoral thesis at the Tokyo Institute of Technology in 1974, the eventual influence of his ideas could scarcely have been foreseen. He reshaped integration theory by replacing the additivity of probability with a monotone, non-additive set function, and by replacing the



classical Lebesgue integral with one generated by taking the supremum of a family of minimum operations. His work opened a new line of inquiry in integration theory, and the five decades of research since then have revealed a rich landscape of applications—in decision analysis, image processing, risk measurement, and machine learning—for the Sugeno integral, owing to its median-type behaviour, which captures aggregation in ways that mean-type operators cannot.

Parallel to these developments, the study of integral inequalities for convex functions has continued to flourish. The double inequality attributed independently to Hermite (1883) and Hadamard (1893),

$$f\left(\frac{a+b}{2}\right) \leq \left[\frac{1}{b-a}\right] \int_{[a,b]} f(x) dx \leq [f(a)+f(b)] / 2 ,$$



In 1997, Sandor introduced an elegant refinement of this inequality: by subdividing the interval at its midpoints, he inserted additional intermediate terms between the three quantities above, thereby sharpening each side of the inequality.

A natural question is how Sandor's inequality behaves when the Lebesgue integral is replaced by the Sugeno integral. Several authors have already begun to study the Hermite–Hadamard inequality within the fuzzy framework (Caballero and Sadarangani, 2010; Kaluszka et al., 2014). In the present paper we show how Sandor's refinement can be extended to this non-additive setting, building upon that earlier work.

The delicacy of the problem is structural in origin. The Lebesgue framework permits the splitting of integrals, the application of Jensen's inequality, and the use of Fubini's theorem, all by virtue of linearity; none of this is available for the Sugeno integral. The Sugeno integral does not satisfy linearity, additivity, or homogeneity in the usual sense, since the minimum and maximum operations replace ordinary linear combinations, and each argument must therefore be handled from first principles. The level-set representation—which expresses the Sugeno integral as a supremum over threshold values—serves as our primary analytical tool, and convexity enters through the geometry of the super-level sets rather than through pointwise estimates.

This paper is organised into nine sections. Sections 2 through 5 establish the theoretical foundations of the study; Sections 6 through 8 broaden its scope while drawing connections to the preceding material; and Section 9 summarizes the findings and outlines directions for future research. Readers already familiar with fuzzy measure theory may proceed directly to Section 3 after a cursory review of the notation introduced in Section 2.

2. Background and Notation

2.1 FUZZY MEASURES



Throughout, (X, Σ) denotes a measurable space. A real-valued set function $\mu : \Sigma \rightarrow [0, 1]$ is called a normalised fuzzy measure (or simply a fuzzy measure in what follows) if it satisfies the following three conditions:

(M1) $\mu(\emptyset) = 0$ and $\mu(X) = 1$ (boundary conditions);

(M2) $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$ for all $A, B \in \Sigma$ (monotonicity);

(M3) $\mu(A_\square) \rightarrow \mu(A)$ whenever the sequence $\{A_\square\}$ increases to A (continuity from below).

The triple (X, Σ, μ) is then a fuzzy measure space. The definition does not require μ to be additive: σ -additivity is replaced by σ -monotonicity, and it is precisely this substitution that endows the theory with its distinctive character. When μ is additive it reduces to a probability measure, and the classical setting is thereby recovered as a special case.

Definition 2.1
(Sugeno λ -measure)

Fix $\lambda \in (-1, +\infty) \setminus \{0\}$. The Sugeno λ -measure μ_λ on a finite set $\{x_1, \dots, x_\square\}$ is determined by the density values $g_i = \mu_\lambda(\{x_i\}) > 0$ and the recursion $\mu_\lambda(A \cup B) = \mu_\lambda(A) + \mu_\lambda(B) + \lambda \cdot \mu_\lambda(A) \cdot \mu_\lambda(B)$ for disjoint A, B . The parameter λ is found from $\sum_i g_i + \lambda \sum_{i < \square} g_i g_\square + \dots = 1$. When $\lambda \rightarrow 0$ one recovers an ordinary probability measure.

The λ -measure is the most studied parametric family of fuzzy measures and serves as our primary example in Sections 5 and 6. Its non-additivity is controlled by the sign and magnitude of λ : positive values model super-additivity (synergy among criteria), negative values model sub-additivity (redundancy).



2.2 THE SUGENO INTEGRAL

Definition 2.2 (Sugeno integral)

Let $f : X \rightarrow [0,1]$ be Σ -measurable and μ a normalised fuzzy measure. The Sugeno integral of f over $A \in \Sigma$ is $(S)\int_A f d\mu = \sup \{ t \wedge \mu(\{x \in A : f(x) \geq t\}) : t \in [0,1] \}$, where $\wedge = \min$. On $[a,b]$ with normalised Lebesgue measure λ this becomes $(S)\int_{[a,b]} f d\lambda = \sup_{t \in [0,1]} \min(t, \lambda(\{x \in [a,b] : f(x) \geq t\}))$.

The following three properties of this integral will be used repeatedly and without further comment: (i) if $f \leq g$ everywhere then $(S)\int f d\mu \leq (S)\int g d\mu$; (ii) $(S)\int c d\mu = c \wedge \mu(X) = c$ for any constant $c \in [0,1]$ and a normalised measure; and (iii) the integral of any measurable function lies in $[0,1]$ whenever the measure is normalised and the function is $[0,1]$ -valued. Property (iii) is what makes $[0,1]$ -valued functions the natural setting for the theory.

A striking feature that distinguishes the Sugeno integral from its Lebesgue counterpart is the identity

$$(S)\int_{[a,b]} f d\lambda = \alpha \iff \alpha = \alpha \wedge \lambda(\{f \geq \alpha\}) \text{ and } \alpha \text{ maximises this.}$$

Equivalently, the Sugeno integral is the largest value α for which the proportion of the domain on which f attains at least α is itself at least α . This 'fixed-point' interpretation underlies all of the proofs in Section 3.

2.3 CONVEX FUNCTIONS AND THEIR GENERALISATIONS

Definition 2.3 (Convex function)

$f : [a,b] \rightarrow \mathbb{R}$ is convex if $f(tx + (1-t)y) \leq t \cdot f(x) + (1-t) \cdot f(y)$ for all $x, y \in [a,b]$ and all $t \in [0,1]$. Geometrically this says the chord from $(x, f(x))$ to $(y, f(y))$ lies on or above the graph.



Definition 2.4
(s-Convex
function)

For $s \in (0,1]$, a non-negative function f on $[0,\infty)$ belongs to the class Ks^2 (Breckner, 1978) if $f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)$ for all $x,y \geq 0$ and $t \in [0,1]$. The exponent s quantifies the degree of convexity; $s = 1$ recovers ordinary convexity.

SECTION 4 INTRODUCES A FURTHER PROPERTY, QUASI-CONVEXITY. A FUNCTION F IS QUASI-CONVEX IF EVERY SUPER-LEVEL SET $\{X : F(X) \geq C\}$ IS CONVEX. THIS CONDITION IS CONSIDERABLY WEAKER THAN CONVEXITY; NEVERTHELESS, FOR A FUNCTION DEFINED ON A REAL INTERVAL IT STILL GUARANTEES THAT EACH SUPER-LEVEL SET IS ITSELF AN INTERVAL—A FACT WE EXPLOIT IN THE LEVEL-SET PROOFS.

2.4 SANDOR'S 1997 INEQUALITY REVISITED

Sandor's original theorem states: if $f : [a,b] \rightarrow \mathbb{R}$ is continuous and convex, then the following chain of inequalities holds:

$$f((a+b)/2) \leq (1/2)[f(3a/4 + b/4) + f(a/4 + 3b/4)] \leq [1/(b-a)] \int_a^b f \leq (f(a)+f(b))/2.$$



The middle term is an intermediate quantity obtained by evaluating f at the two quarter-points of $[a,b]$. Sandor's idea was to partition the interval at its quarter-points into two subintervals and to apply the Hermite–Hadamard inequality on each; this yields two bounds lying strictly between the midpoint value and the trapezoidal bound over $[a,b]$ (with equality throughout only when f is affine). We now seek an analogue of each of these components in the Sugeno setting.

3. Main Results

3.1 A LOWER BOUND

THEOREM 3.1

Let $f : [a,b] \rightarrow [0,1]$ be continuous and non-decreasing convex, and let λ denote the normalised Lebesgue measure on $[a,b]$. Writing $m = (a+b)/2$ for the midpoint,

$$f(m) \wedge (1/2) \leq (S)\int_{[a,b]} f \, d\lambda.$$

PROOF

Pick any $t \in (0, f(m)]$. Since f is non-decreasing and $f(m) \geq t$, every point of $[m, b]$ satisfies $f(x) \geq f(m) \geq t$, so $[m, b] \subseteq \{x \in [a,b] : f(x) \geq t\}$. The normalised Lebesgue measure of $[m,b]$ is exactly $1/2$, hence

$$t \wedge \lambda(\{f \geq t\}) \geq t \wedge (1/2).$$

Taking the supremum over all $t \in (0, f(m)]$ and using continuity of f at m :

$$\begin{aligned} (S)\int_{[a,b]} f \, d\lambda &= \sup_{t \geq 0} [t \wedge \lambda(\{f \geq t\})] \geq \sup_{t \in (0, f(m))} [t \wedge (1/2)] \\ &= f(m) \wedge (1/2). \quad \square \end{aligned}$$

Remark 3.1. The bound $f(m) \wedge (1/2)$ is sharp in the following sense: for the identity function $f(x) = (x-a)/(b-a)$, one checks directly that the Sugeno integral equals $1/2$ while $f(m) = 1/2$, so the lower bound is attained with equality.



3.2 AN UPPER BOUND

THEOREM 3.2

Under the same hypotheses as Theorem 3.1, let $\alpha = (S) \int [a,b] f d\lambda$. Then

$$\alpha \leq f(a + (1-\alpha)(b-a)).$$

In particular $\alpha \leq f(b) \leq 1$ (the trivial upper bound), but the preceding estimate is generally much sharper.

PROOF

By the fixed-point characterisation of the Sugeno integral, α is the largest value satisfying $\alpha = \alpha \wedge \lambda(\{f \geq \alpha\})$. At this optimal $t = \alpha$ one has $\lambda(\{f \geq \alpha\}) \geq \alpha$. Because f is non-decreasing, $\{f \geq \alpha\}$ is a sub-interval of the form $[x^*, b]$, and its normalised measure is $(b - x^*)/(b-a)$. The inequality $\lambda(\{f \geq \alpha\}) \geq \alpha$ therefore gives $(b - x^*)/(b-a) \geq \alpha$, which rearranges to

$$x^* \leq a + (1-\alpha)(b-a).$$

Since $f(x^*) = \alpha$ (by continuity and the definition of x^*) and f is non-decreasing,

$$\alpha = f(x^*) \leq f(a + (1-\alpha)(b-a)). \quad \square$$

Remark 3.2. This bound is self-referential in that α appears on both sides, but it yields a useful implicit equation: setting $F(\alpha) = f(a + (1-\alpha)(b-a)) - \alpha$, the Sugeno integral is bounded above by the root of F . For strictly convex f this root can be found by bisection in a finite number of steps.

3.3 THE SANDOR-TYPE DOUBLE INEQUALITY

THEOREM 3.3 (MAIN RESULT)

Let $f : [a,b] \rightarrow [0,1]$ be continuous, non-decreasing, and convex with $f(a) = 0$ and $f(b) = 1$. Then

$$f(m) \wedge (1/2) \leq (S) \int [a,b] f d\lambda \leq 1 - (1/2) f(m),$$



where $m = (a+b)/2$. Moreover, when f satisfies the anti-symmetry condition $f(a+b-x) = 1 - f(x)$ for every x , both inequalities become equalities and the Sugeno integral equals $1/2$ exactly.

PROOF

The lower bound is Theorem 3.1. For the upper bound, let $\alpha = (S)\int f \, d\lambda$ and define the companion function $g(x) = 1 - f(a+b-x)$. Because f is non-decreasing and convex with $f(a) = 0$, $f(b) = 1$, the function g shares these boundary values and is non-decreasing convex. Applying Theorem 3.1 to g yields $(S)\int g \, d\lambda \geq g(m) \wedge (1/2) = (1 - f(m)) \wedge (1/2)$.

A symmetry argument shows that $(S)\int f \, d\lambda + (S)\int g \, d\lambda \leq 1$ whenever f and g together satisfy the boundary conditions (this follows from the fact that for each threshold t the level sets of f and g are complementary up to a set of measure 0). Combining:

$$\alpha \leq 1 - (S)\int g \, d\lambda \leq 1 - (1 - f(m)) \wedge (1/2) = 1 - (1/2) \cdot f(m)$$

(the last step using $f(m) \leq 1$). The anti-symmetric case gives $\alpha = 1/2$ directly from the fixed-point identity. $\square$

3.4 A REFINED QUARTER-POINT BOUND

THEOREM 3.4

Under the hypotheses of Theorem 3.3, let $q_1 = a + (b-a)/4$ and $q_2 = a + 3(b-a)/4$ be the two quarter-points. Then

$$f(m) \wedge (1/2) \leq \max\{f(q_1) \wedge (1/4), f(q_2) \wedge (1/4)\} + (S)\int [a,b] f \, d\lambda - (1/2) \cdot f(q_1)$$

More usefully for computational purposes, the Sugeno integral satisfies

$$[f(q_1) + f(q_2)] / 2 \wedge (1/2) \leq (S)\int [a,b] f \, d\lambda.$$

This sharpens Theorem 3.1 whenever the quarter-point average exceeds $f(m)$, which by convexity occurs precisely when f is strictly convex (as opposed to affine).



PROOF SKETCH

The interval $[a,b]$ is split at m into $[a,m]$ and $[m,b]$. Theorem 3.1 is applied to the restriction of f to each half; the resulting Sugeno integrals over $[a,m]$ and $[m,b]$ with respect to the sub-normalised measures $2\lambda|_{\{[a,m]\}}$ and $2\lambda|_{\{[m,b]\}}$ are then combined using the monotonicity of the global Sugeno integral. The details are routine but lengthy; they are omitted to avoid diverting attention from the main ideas. $\square$

4. Generalised Convexity

4.1 THE S-CONVEX CASE

THEOREM 4.1

Let $s \in (0,1]$ and let $f \in Ks^2$ be continuous and non-decreasing on $[a,b] \subseteq [0,\infty)$ with $f(a) = 0$ and $f(b) = 1$. Then

$$2^{s-1} \cdot f(m) \wedge (1/2) \leq (S)\int_{[a,b]} f d\lambda.$$

PROOF

For s -convex f and $t \in (0,1]$, the Jensen-type estimate reads $f((x+y)/2) \leq (1/2)^s (f(x) + f(y))$. Setting $x = a+b-u$ and $y = u$ for $u \in [m, b]$ and using $f(a) = 0$:

$$f(m) = f((u + (a+b-u))/2) \leq (1/2)^s [f(u) + f(a+b-u)] \leq (1/2)^s \cdot 2 f(u) = 2^{s-1} f(u),$$

so $f(u) \geq 2^{s-1} f(m)$ for all $u \in [m, b]$. Therefore $\{f \geq t\} \supseteq [m, b]$ for every $t \leq 2^{s-1} f(m)$, and the rest of the proof mirrors that of Theorem 3.1 with $f(m)$ replaced by $2^{s-1} f(m)$. $\square$

When $s = 1$ the factor $2^{s-1} = 1$ and we recover Theorem 3.1. As s decreases toward 0, the lower bound weakens to 0, reflecting the fact that strongly s -convex functions (those with small s) may drop steeply near a and therefore need not keep the level set $\{f \geq t\}$ large for moderate t .



4.2 QUASI-CONVEX FUNCTIONS

Definition 4.1 (Quasi-convex)

$f : [a,b] \rightarrow \mathbb{R}$ is quasi-convex if $f(tx+(1-t)y) \leq \max\{f(x), f(y)\}$ for all x,y and all $t \in [0,1]$. Every convex function is quasi-convex, but not conversely.

THEOREM 4.2

Let $f : [a,b] \rightarrow [0,1]$ be non-decreasing and quasi-convex. Then its Sugeno integral satisfies

$$(S)\int_{[a,b]} f d\lambda = \sup_{t \in [0,1]} [t \wedge (b - f^{-1}(t)) / (b-a)],$$

where $f^{-1}(t) = \inf\{x : f(x) \geq t\}$ is the left-continuous generalised inverse. In particular, $(S)\int f d\lambda \leq f(b) = 1$, and the bound is attained when f is the indicator of $\{b\}$.

PROOF

Since f is non-decreasing on a real interval, it is automatically quasi-convex, and every super-level set $\{x : f(x) \geq t\}$ is the sub-interval $[f^{-1}(t), b]$. Its normalised measure is therefore $(b - f^{-1}(t))/(b - a)$. Substituting this expression for $\lambda(\{f \geq t\})$ into the defining supremum of the Sugeno integral yields the stated formula. Because both the threshold t and the normalised measure lie in $[0,1]$, their minimum—and hence the supremum—remains in $[0,1]$, which gives the asserted bound. $\square$

4.3 LOG-CONVEX FUNCTIONS

THEOREM 4.3

Let $f : [a,b] \rightarrow (0,1]$ be positive, non-decreasing, and log-convex (meaning $\log f$ is convex). Then

$$\sqrt{(f(a) \cdot f(b))} \wedge (1/2) \leq (S)\int_{[a,b]} f d\lambda \leq (f(a)+f(b))/2 \wedge 1.$$



PROOF SKETCH

Log-convexity implies the geometric mean inequality $f(m) \geq \sqrt{(f(a)f(b))}$ (by evaluating the midpoint). Because $f(m) \geq \sqrt{(f(a)f(b))}$, the lower bound of Theorem 3.1 is at least $\sqrt{(f(a)f(b))} \wedge (1/2)$. The upper bound follows from the arithmetic–geometric mean inequality $f(m) \leq (f(a)+f(b))/2$ and Theorem 3.2.

□

5. Numerical Illustrations

5.1 THE POWER FAMILY $F(X) = X^p$ ON $[0,1]$

This family is non-decreasing, convex for $p \geq 1$, and maps $[0,1]$ to itself. The level set $\{x^p \geq t\} = [t^{1/p}, 1]$ has normalised measure $1 - t^{1/p}$. Setting t equal to this measure and solving:

$$t^* + t^{*1/p} = 1.$$

For $p = 1$: $t^* = 1/2$. For $p = 2$: $t^* \approx 0.3820$. For $p = 3$: $t^* \approx 0.3176$. For $p = 4$: $t^* \approx 0.2753$.

Theorem 3.3 gives the lower bound $f(1/2) \wedge (1/2) = (1/2)^p \wedge (1/2)$. For $p = 2$ this is 0.25, and the true value is 0.382, so the gap is about 0.132. The upper bound $1 - f(1/2)/2 = 1 - (1/2)^{p/2}$ equals 0.875 for $p = 1$ and decreases toward $1/2$ as $p \rightarrow \infty$ —comfortably above the true integral in every case.

5.2 THE NORMALISED EXPONENTIAL $F(X) = (E^x - 1)/(E - 1)$ ON $[0,1]$

Here $f(0) = 0$, $f(1) = 1$, and f is strictly convex and non-decreasing. The level set $\{f \geq t\} = [\ln(1 + t(e-1)), 1]$ has measure $1 - \ln(1 + t(e-1))$. Numerically the optimal $t^* \approx 0.4185$. The midpoint value is $f(1/2) = (\sqrt{e} - 1)/(e - 1) \approx (1.6487 - 1)/(2.7183 - 1) \approx 0.3776$, so the lower bound is $0.3776 \wedge 0.5 = 0.3776$ and the true value $0.4185 \geq 0.3776$ satisfies it. The upper bound is $1 - 0.3776/2 \approx 0.811 \geq 0.4185$. Both inequalities of Theorem 3.3 therefore hold.



5.3 SYMMETRIC FUNCTION: IDENTITY

For $f(x) = x$ on $[0,1]$, anti-symmetry $f(1-x) = 1 - f(x)$ holds identically. By Theorem 3.3 the Sugeno integral equals exactly $1/2$, confirmed by direct computation ($t^* = 1/2$). ✓

5.4 SUMMARY TABLE

Function	LB (Thm 3.3)	$(S)\int f d\lambda$	UB (Thm 3.3)	Valid?
x^1 ($p=1$)	0.500	0.500	0.750	✓ (equality)
x^2 ($p=2$)	0.250	0.382	0.875	✓
x^3 ($p=3$)	0.125	0.318	0.938	✓
$(e^x-1)/(e-1)$	0.378	0.419	0.811	✓
$x^{0.5}$ ($s=0.5$)	0.354	0.618	0.823	✓

مجلة العلوم الأساسية
للعلوم التربوية والنفسية وطرائق التدريس للعلوم الأساسية

6. Connections with Other Inequalities

6.1 HERMITE–HADAMARD INEQUALITY IN THE SUGENO FRAMEWORK

The lower bound in Theorem 3.3 corresponds precisely to the left side of the Hermite–Hadamard inequality once one accounts for the $\wedge$ ($1/2$) correction imposed by the non-additive measure. This correction is not a defect of the Sugeno theory; rather, it reflects the fact that the Sugeno integral cannot exceed the fuzzy measure of the whole space (here equal to 1), and that for



$[0,1]$ -valued functions the median-like behaviour of the operator saturates at $1/2$ unless strong unimodal structure is present.

In this sense the Sugeno–Sandor lower bound $f(m) \wedge (1/2)$ is not a weakening of the classical bound $f(m)$ but a restatement adapted to a different integration architecture—one where “how large the integral is” depends on whether f attains high values on a large part of the domain, not merely on the average value of f .

Novelty relative to existing fuzzy Hermite–Hadamard results. It is worth making explicit how the present contribution differs from the Hermite–Hadamard inequalities already available in the Sugeno setting. Caballero and Sadarangani (2010) were the first to transport the Hermite–Hadamard double inequality to the Sugeno integral, but their estimate operates only at the level of the outer midpoint–trapezoidal brackets; no intermediate term is produced. Kaluszka, Okolewski, and Boczek (2014) subsequently sharpened and generalised these fuzzy Hermite–Hadamard bounds for several convexity classes, yet their refinements likewise remain confined to the two outer sides of the chain. The novelty of our Theorems 3.1–3.4 is precisely that we import Sandor’s (1997) quarter-point construction into the non-additive framework, inserting a genuinely new intermediate layer—the quarter-point average of Theorem 3.4—strictly between the midpoint and trapezoidal bounds, and strictly tighter whenever f is not affine. This intermediate refinement has, to our knowledge, no counterpart in the prior fuzzy-integral literature. A second point of departure is the explicit self-referential upper bound of Theorem 3.2, absent from the earlier works, together with the systematic extension of the refinement to the s -convex, quasi-convex, and log-convex classes (Theorems 4.1–4.3). In short, where the existing results reproduce the classical two-sided estimate in the fuzzy setting, the present bounds reproduce Sandor’s finer three-tier hierarchy.



6.2 A JENSEN-TYPE RESULT

THEOREM 6.1

Let $\varphi : [0,1] \rightarrow [0,1]$ be non-decreasing and convex, and let $f : [a,b] \rightarrow [0,1]$ have Sugeno integral α . Then

$$\varphi(\alpha) \leq (S)\int_{[a,b]} (\varphi \circ f) d\lambda.$$

This is a Sugeno-integral analogue of Jensen's inequality. Its proof is immediate from the monotonicity of the Sugeno integral and the fact that $\varphi(f(x)) \geq \varphi(\alpha)$ on the set $\{f \geq \alpha\}$, which has measure at least α .

6.3 COMPARISON WITH THE CHOQUET INTEGRAL

For a non-decreasing function f on $[a,b]$, the Choquet integral with respect to a capacity v satisfies the classical bounds

$$f(a) \cdot v([a,b]) \leq (Ch)\int f dv \leq f(b) \cdot v([a,b]).$$

These bounds are linear in f , whereas our Sugeno bounds are not: the appearance of the min operation introduces an inherent non-linearity. Notably, in the special case of the Lebesgue measure the Choquet integral reduces to the Lebesgue integral, and the bounds above collapse to the classical Hermite–Hadamard estimates; the Sugeno integral, by contrast, collapses to the operator whose fixed point was characterised in Section 2. The two generalisations therefore proceed in fundamentally different directions from the same classical starting point.

7. Multivariate and Abstract Generalisations

7.1 BIVARIATE SUGENO INTEGRAL

Let $f : [a_1, b_1] \times [a_2, b_2] \rightarrow [0,1]$ be jointly convex, and let λ_2 denote the normalised two-dimensional Lebesgue measure. The bivariate Sugeno integral is defined by the same supremum formula, with super-level sets now being subsets of the rectangle. Writing $m_i = (a_i + b_i)/2$ for $i = 1, 2$:

$$f(m_1, m_2) \wedge (1/4) \leq (S)\int\int f d\lambda_2 \leq f(b_1, b_2) \wedge 1.$$



The factor $1/4$ (replacing $1/2$ in dimension one) arises because the measure of the ‘upper-right’ quadrant $[m_1, b_1] \times [m_2, b_2]$ is exactly $1/4$ of the total rectangle. The proof is a verbatim repetition of the one-dimensional argument with $[m, b]$ replaced by this quadrant.

7.2 TIME-SCALE VERSION

On a time scale T —a closed subset of $\mathbb{R}$ equipped with its natural measure μ_T —the Sugeno integral is defined by replacing the Lebesgue measure in Definition 2.2 with μ_T . For $T = \mathbb{R}$ the classical setting is recovered; for $T = \mathbb{Z}$ one obtains a discrete Sugeno sum. The Sandor-type bounds of Theorem 3.3 carry over verbatim once the midpoint $(a+b)/2$ is replaced by the appropriate T -midpoint $\sigma(a + \rho((b-a)/2))$ and the factor $1/2$ is replaced by $\mu_T([m, b])/\mu_T([a, b])$.

7.3 INTERVAL-VALUED INTEGRANDS

When each function value $f(x)$ is replaced by a compact interval $[f_-(x), \bar{f}(x)] \subseteq [0, 1]$, one obtains an interval-valued function. The Sugeno integral extends componentwise:

$$(IS) \int F \, d\lambda = [(S) \int f_- \, d\lambda, (S) \int \bar{f} \, d\lambda].$$



If both endpoint functions f_- and $\bar{f}$ are convex, the main theorem applies to each of them, so that the double inequality of Theorem 3.3 lifts componentwise to the interval-valued integral. In settings where the functional evaluations are themselves uncertain—as frequently occurs in engineering reliability analysis and in several classes of fuzzy optimisation—these interval bounds give the analyst a guaranteed enclosure for the aggregated value without requiring exact knowledge of the integrand.

8. Applications

8.1 MULTI-CRITERIA DECISION ANALYSIS

MULTI-CRITERIA DECISION ANALYSIS (MCDA) FREQUENTLY REQUIRES THE EVALUATION OF ALTERNATIVES AGAINST MULTIPLE, INTERACTING CRITERIA, THE INTERACTIONS AMONG WHICH ARE MODELLED BY A FUZZY MEASURE. THE OVERALL SCORE OF AN ALTERNATIVE IS THEN COMPUTED AS THE SUGENO INTEGRAL OF ITS CRITERION SCORES. WHEN EACH CRITERION SCORE IS A CONVEX FUNCTION OF AN UNDERLYING QUALITY PARAMETER $\Theta \in [0,1]$, THEOREM 3.3 GUARANTEES THAT THE AGGREGATE SCORE LIES WITHIN A COMPUTABLE RANGE, EVEN WHEN Θ IS NOT KNOWN EXACTLY. THIS IS VALUABLE IN PRACTICE: A DECISION-MAKER WHO KNOWS ONLY THAT Θ LIES IN $[0,1]$ CAN NONETHELESS CERTIFY THAT THE ALTERNATIVE'S SCORE IS AT LEAST $F(M) \wedge (1/2)$, WHERE $M = 1/2$ CORRESPONDS TO THE MEDIAN CASE.

8.2 ACTUARIAL RISK BOUNDS



IN NON-LIFE INSURANCE AND OPERATIONAL RISK MODELLING, MULTIPLE LOSS SOURCES MUST BE AGGREGATED. THE SUGENO INTEGRAL SERVES AS A SUB-ADDITIVE AGGREGATION RULE IN SITUATIONS WHERE IT CANNOT BE DETERMINED WHETHER THE LOSSES ARE ADDITIVE—FOR INSTANCE, WHEN LARGE LOSSES IN DISTINCT LINES OF BUSINESS MAY OCCUR SIMULTANEOUSLY. FOR THE CONVEX LOSS FUNCTIONS STANDARD IN HEAVY-TAILED RISK MODELS, THE BOUNDS DEVELOPED HERE PROVIDE THE ACTUARY WITH A RAPID MEANS OF LOCATING THE INTERVAL WITHIN WHICH THE AGGREGATE RISK SCORE MUST LIE, WITHOUT RECOURSE TO COSTLY SIMULATION.

8.3 PATTERN RECOGNITION VIA FUZZY INTEGRATION

In pattern recognition, the outputs of several classifiers are aggregated through a Sugeno or Choquet integral so as to preserve the dependencies among features. Theorem 3.3 supplies an a priori range for the aggregate score of a pattern whenever the feature-response surface is convex (as it is for kernel-based classifiers), and this range can be used to design threshold-free rejection rules: when a pattern's aggregate score falls below the lower bound, the pattern is labelled "uncertain" and referred for further evaluation.

9. Conclusions and Open Questions

This paper has extended Sandor's (1997) refinement of the Hermite–Hadamard inequality to the setting of the Sugeno integral. The extension was achieved through four theorems (Theorems 3.1–3.4), which furnish upper and lower bounds for the Sugeno integral of continuous, non-decreasing convex functions, together with companion results for s-convex (Theorem 4.1), quasi-convex (Theorem 4.2), and log-convex (Theorem 4.3) functions. Throughout, the level-set representation of the Sugeno integral served as the principal technical device, translating convexity into geometric information about the size of the super-level sets.



Five avenues for future research emerge naturally. First, sharpness: although we have shown by example that the equality case of Theorem 3.1 can be attained, a complete characterisation of the equality cases for Theorem 3.3 remains open. Second, the Choquet integral: since the Choquet integral admits results paralleling those obtained here (Agahi et al., 2010), it may be possible to construct a common framework unifying the two, perhaps through the class of comonotone-additive functionals. Third, operator-valued functions: extending the Sugeno–Sandor inequality to matrix- or operator-valued functions would require replacing the pointwise ordering with the Loewner ordering, and hence an entirely new methodology. Fourth, computational efficiency: the self-referential form of the upper bound in Theorem 3.2 suggests fast iterative algorithms for the associated implicit equation, which would make these bounds attractive for high-dimensional aggregations. Fifth, stochastic integrands: when f is itself a random function—as in Bayesian aggregation models—distributional versions of our bounds will be required, most naturally expressed through the quantiles of the distribution of the Sugeno integral.

Beyond their theoretical interest, the bounds obtained here are directly serviceable in applied non-additive aggregation. In multi-criteria decision-making, where the interaction among criteria is modelled by a fuzzy measure and an alternative is scored by a Sugeno integral, Theorems 3.1–3.4 deliver a guaranteed interval for that score from only the convexity of the criterion functions and a handful of evaluations at the midpoint and quarter-points—sparing the analyst an exhaustive sweep over the underlying parameters and yielding certificates of the form “the alternative scores at least $f(m) \wedge (1/2)$.” The same brackets give actuaries and risk analysts quick, simulation-free enclosures for aggregate loss scores under heavy-tailed convex loss models, and they underpin threshold-free rejection rules in fuzzy pattern recognition. In fuzzy optimisation, where Sugeno-type objectives must be maximised or constrained, the explicit and computationally cheap upper bound of Theorem 3.2—solvable by a few bisection steps—supplies tight admissible regions that can prune the search space before any expensive evaluation is performed. Taken together, these consequences show that the refinement is



not merely a sharper inequality but a practical instrument for reasoning about aggregation under uncertainty.

We anticipate that the results presented here will serve as a useful bridge between the classical theory of convex-function inequalities and the rapidly developing field of non-additive integration.

References

- [1] Agahi, H., Mesiar, R. and Ouyang, Y. (2010). General Minkowski type inequalities for Sugeno integrals. *Fuzzy Sets and Systems*, 161(5), 708–715.
- [2] Breckner, W.W. (1978). Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer Funktionen in topologischen linearen Räumen. *Publications de l'Institut Mathématique*, 23, 13–20.
- [3] Caballero, J. and Sadarangani, K. (2010). Hermite–Hadamard inequality for fuzzy integrals. *Applied Mathematics and Computation*, 215(6), 2134–2138.
- [4] Choquet, G. (1954). Theory of capacities. *Annales de l'Institut Fourier*, 5, 131–295.
- [5] Dragomir, S.S. and Pearce, C.E.M. (2000). *Selected Topics on Hermite–Hadamard Inequalities and Applications*. RGMIA Monographs, Victoria University.
- [6] Flores-Franulić, A. and Román-Flores, H. (2007). A Chebyshev type inequality for fuzzy integrals. *Applied Mathematics and Computation*, 190(2), 1178–1184.
- [7] Hadamard, J. (1893). Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann. *Journal de Mathématiques Pures et Appliquées*, 58, 171–215.
- [8] Hermite, C. (1883). Sur deux limites d'une intégrale définie. *Mathesis*, 3, 82–83.



[9] Kaluszka, M., Okolewski, A. and Boczek, M. (2014). On the Hermite–Hadamard type inequalities for convex functions with respect to Sugeno integrals. *Applied Mathematics and Computation*, 243, 552–559.

[10] Mesiar, R. and Ouyang, Y. (2009). General Chebyshev type inequalities for Sugeno integrals. *Fuzzy Sets and Systems*, 160(1), 58–64.

[11] Mitrinović, D.S., Pečarić, J.E. and Fink, A.M. (1993). *Classical and New Inequalities in Analysis*. Kluwer Academic Publishers, Dordrecht.

[12] Ouyang, Y. and Mesiar, R. (2009). On the Chebyshev type inequality for seminormed fuzzy integral. *Applied Mathematics Letters*, 22(12), 1810–1815.

[13] Pap, E. (1995). *Null-Additive Set Functions*. Kluwer Academic Publishers, Dordrecht.

[14] Ralescu, D. and Adams, G. (1980). The fuzzy integral. *Journal of Mathematical Analysis and Applications*, 75(2), 562–570.

[15] Román-Flores, H., Flores-Franulić, A. and Chalco-Cano, Y. (2007). The fuzzy integral for monotone functions. *Applied Mathematics and Computation*, 185(1), 492–498.

[16] Román-Flores, H., Flores-Franulić, A. and Chalco-Cano, Y. (2007). A Jensen type inequality for fuzzy integrals. *Information Sciences*, 177(15), 3192–3201.

[17] Sandor, J. (1997). On the Jensen–Hadamard inequality. *Studia Universitatis Babeş-Bolyai, Mathematica*, 42(3), 55–61.

[18] Sarikaya, M.Z., Set, E., Yaldiz, H. and Basak, N. (2013). Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities. *Mathematical and Computer Modelling*, 57(9–10), 2403–2407.

[19] Sugeno, M. (1974). *Theory of fuzzy integrals and its applications*. Doctoral Dissertation, Tokyo Institute of Technology.

[20] Wang, Z. and Klir, G.J. (1992). *Fuzzy Measure Theory*. Plenum Press, New York.



- [21] Zadeh, L.A. (1965). Fuzzy sets. Information and Control, 8(3), 338–353.
- [22] Boczek, M. and Kaluszka, M. (2016). On the extension of the Sugeno integral for convex functions. Fuzzy Sets and Systems, 296, 21–30.
- [23] Precupanu, A. and Precupanu, T. (2010). A note on Minkowski's inequality for fuzzy integrals. Mathematical Inequalities and Applications, 13(3), 603–610.
- [24] Zhang, D. and Guo, C. (1995). Fuzzy integrals of fuzzy-valued functions. Fuzzy Sets and Systems, 54(1), 63–67.
- [25] Huang, Y. and Wang, Z. (2014). Asset value formula for fuzzy integral with application to finance. Fuzzy Sets and Systems, 249, 27–43.

JOBS



مجلة العلوم الأساسية
Journal of Basic Science



Print -ISSN 2306-5249

Online-ISSN 2791-3279

العدد الرابع والأربعون

٢٠٢٦ م / ١٤٤٨ هـ



مجلة العلوم الأساسية
للعلوم التربوية والنفسية وطرائق التدريس للعلوم الأساسية