JOBS مجلة العلوم الأساسية
Journal of Basic Science
Print -ISSN 2306-5249
Online-ISSN 2791-3279
العدد السادس والأربعون
٢٠٢٦م/١٤٤٨هـ
العدد السادس والأربعون
(496)- (485)

اختبارات المعادلات المحدودة للفئات الفرعية النقية والتماسك الأيمن للمونويدات

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المستخلص:

يقدّم هذا البحث إجراءً ذا نواة منتهية لاستخراج عروض الأفعال الفرعية المنتهية التوليد داخل الأفعال اليمنى المنتهية العرض. وينشئ، لكل متجه منتهٍ من المولدات، تطبيقًا قانونيًا من الفعل الحر وتوافق نواته. ونثبت أن الفعل الفرعي منتهي العرض إذا وفقط إذا كان توافق النواة منتهي التوليد، ومن ذلك نستنتج توصيفًا بالنواة المنتهية للمونويدات المتماسكة يمينيًا. كما نصوغ اختبارات معادلات منتهية للنقاوة، ومعيارًا لإعادة الكتابة يستخرج العلاقات المعرِّفة. ويعرض البحث نموذجًا عدديًا لحالات صيانة المعدات تُحسب فيه النواة والعرض المنتهي ومجموعا المخاطر قبل الصيانة وبعدها. وبذلك تتحول عبارات التماسك المجردة إلى إجراء تحقق صريح.

الكلمات المفتاحية: مونويد؛ فعل يميني؛ فعل فرعي؛ مونويد متماسك يمينيًا؛ نقاوة؛ عرض منتهٍ؛ توافق النواة؛ نظام إعادة كتابة

FINITE EQUATION TESTS FOR PURE SUB-ACTS AND RIGHT COHERENCY OF MONOIDS

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ABSTRACT

This paper gives a finite-kernel procedure for extracting presentations of finitely generated sub-acts of finitely presented right acts. For a tuple $\mathbf{b} = (b_1, \dots, b_m)$, we use the canonical map $\pi_{\mathbf{b}}: F_m \to A$ and its kernel congruence $K_A(\mathbf{b})$. We prove that $B = \langle b_1, \dots, b_m \rangle_S$ is finitely presented if and only if $K_A(\mathbf{b})$ is finitely generated, and we derive a finite-kernel characterization of right coherent monoids. We also formulate finite equation tests for purity and a rewriting criterion for extracting defining relations. A numerical maintenance-state model computes the kernel, a finite presentation, and the risk totals before and after maintenance. The results turn abstract coherency statements into an explicit verification procedure.

Keywords: monoid; right act; sub-act; right coherent monoid; purity; finite presentation; kernel congruence; rewriting system.

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1. INTRODUCTION

Let (S, 1) be a monoid. A right S-act is a nonempty set A, equipped with an action A×S→A, (a,s)↦as, which has the following properties:

$$a1 = a, \qquad a(st) = (as)t$$

Yet finite generation and finite presentation are severe natural structural conditions for monoids that are non-additive versions of right modules. The monoid M is called right coherent if every finitely generated sub-act of every finitely presented right S-act is finitely presented (Gould, 1992). This field of mathematics now brings together finite systems of equations, purity, congruence finiteness, covers, flatness, and rewriting.

Recent contributions include the purity characterization of coherency (Yang & Gould, 2023), separability and congruence-chain conditions for acts (Miller, 2022; Khosravi & Roueentan, 2024), injectivity and factorization methods (Stepanova, 2023; Li & Zhu, 2023), covers and exact flatness (Zhang & Zhao, 2024; Rasouli & Aminizadeh, 2025), and coherency results for weak, transformation, and partition settings (Brookes et al., 2025; Dasar et al., 2025). Rewriting methods supply a complementary route to explicit relations (Meha, 2023).

The practical difficulty is that coherency guarantees a finite presentation for a generated sub-act but does not automatically display its relations. Suppose

$$A = \langle X \mid R \rangle_S, \qquad B = \langle b_1, \dots, b_m \rangle_S \leq A.$$

The procedure developed below starts from the canonical map $F_m \to B$, computes its kernel congruence, and reduces finite presentability to finite generation of that kernel. The same language also records finite obstructions to purity and the relation-extraction step in rewriting arguments. Section 7 gives a fully worked numerical model.

2. PRELIMINARIES AND MOTIVATING EXAMPLES

Throughout the paper, S denotes a monoid with identity 1, and all acts are nonempty right S-acts (Kilp et al., 2000). For $Y \subseteq A$, write $\langle Y \rangle_S = YS$ for the sub-act generated by Y. The free right act on a set X is

$$F_X = X \times S, \qquad (x,s)t = (x,st).$$

A right congruence ρ on F_X is an equivalence relation such that $(u,v) \in \rho$ implies $(ut,vt) \in \rho$ for every $t \in S$. If $R \subseteq F_X \times F_X$, then $\langle R \rangle$ denotes the right congruence generated by R, and

$$A = \langle X \mid R \rangle_S \quad \text{means} \quad A \cong F_X/\langle R \rangle.$$

The act is finitely presented when both X and R are finite (Miller & Ruškuc, 2019).

Definition 2.1. A monoid S is **right coherent** if every finitely generated sub-act of every finitely presented right S-act is finitely presented.

Definition 2.2. A monoid S is **weakly right coherent** if every finitely generated right ideal of S is finitely presented as a right S-act.

Example 2.1. Let $S = \{1, m\}$ with $m^2 = m$. Since S is finite, Theorem 5.1 shows that it is right coherent, and hence weakly right coherent by Proposition 5.2. Its principal right ideal $mS = \{m\}$ has the finite presentation

$$mS \cong \langle e \mid em = e \rangle_S.$$

Let $B \leq A$. A finite system of equations over B consists of equations in variables $x_1, \ldots, x_n$ of the forms

$$x_i s = x_j t, \qquad x_i s = b,$$

where $s, t \in S$ and $b \in B$.

Definition 2.3. The sub-act B is **equationally pure** in A if every finite system of equations over B that has a solution in A also has a solution in B (Yang & Gould, 2023).

Example 2.2. Let $S = \langle t \rangle = \{1, t, t^2, \ldots\}$ be the free monoid on one generator, and let $A = \{a, b, c\}$ with $at = b$, $bt = c$, and $ct = c$. The sub-act $C = \{c\}$ is pure because the map $r: A \to C$, $r(a) = r(b) = r(c) = c$, is a retraction. In contrast, $B = \{b, c\}$ is not pure: the equation $xt = b$ has the solution $x = a$ in A, but no solution in B. Thus, this single equation is a finite purity obstruction.

3. THE FINITE-KERNEL METHOD

Let A be a right S-act and let $\mathbf{b} = (b_1, \dots, b_m) \in A^m$. Put $F_m = \{1, \dots, m\} \times S$ and define

$$\pi_{\mathbf{b}}: F_m \to A, \qquad \pi_{\mathbf{b}}(i, s) = b_i s.$$

Its image is $B = \langle b_1, \dots, b_m \rangle_S$, and its kernel congruence is

$$K_A(\mathbf{b}) = \ker \pi_{\mathbf{b}} = \{\big((i, s), (j, t)\big) \in F_m \times F_m : b_i s = b_j t\}.$$

Hence the quotient theorem gives

$$F_m / K_A(\mathbf{b}) \cong B.$$

Theorem 3.1 (Generator-kernel criterion). The sub-act $B = \langle b_1, \dots, b_m \rangle_S$ is finitely presented if and only if $K_A(\mathbf{b})$ is finitely generated as a right congruence on F_m.

Proof. Suppose first that $K_A(\mathbf{b}) = \langle \Omega \rangle$ for a finite set $\Omega \subseteq F_m \times F_m$. Then

$$B \cong F_m / K_A(\mathbf{b}) = F_m / \langle \Omega \rangle,$$

so B has the finite presentation $\langle e_1, \dots, e_m \mid \Omega \rangle_S$.

For the reverse implication, assume that B is finitely presented. Since $b_1, \dots, b_m$ generate B, finite Tietze transformations give a finite set $\Omega \subseteq F_m \times F_m$ and an isomorphism

$$\varphi: F_m / \langle \Omega \rangle \to B$$

that sends the class of $(i, 1)$ to b_i (Miller & Ruškuc, 2019). If $q: F_m \to F_m / \langle \Omega \rangle$ is the quotient map, then $\pi_{\mathbf{b}} = \varphi \circ q$. Because φ is injective,

$$K_A(\mathbf{b}) = \ker \pi_{\mathbf{b}} = \ker q = \langle \Omega \rangle.$$

Thus, the kernel congruence is generated by the finite set Ω.

Now let $A = \langle X \mid R \rangle_S$ be finitely presented. For a finite tuple of terms $\boldsymbol{\tau} = (\tau_1, \dots, \tau_m)$ in F_X, define

$$K_A(\boldsymbol{\tau}) = \{\big((i, u), (j, v)\big) \in F_m \times F_m : [\tau_i u]_A = [\tau_j v]_A\}.$$

Theorem 3.2 (Finite-kernel characterization). For a monoid S, the following conditions are equivalent:

(i) S is right coherent.

(ii) For every finite presentation $A = \langle X \mid R \rangle_S$ and every finite tuple $\boldsymbol{\tau}$ of terms in F_X, the congruence $K_A(\boldsymbol{\tau})$ is finitely generated.

Proof. Assume (i). The sub-act generated by $[\tau_1]_A, \dots, [\tau_m]_A$ is finitely generated inside the finitely presented act A, so it is finitely presented. Theorem 3.1 makes $K_A(\boldsymbol{\tau})$ finitely generated.

For the reverse implication, assume (ii), let A be finitely presented, and let $B \le A$ be generated by $b_1, \dots, b_m$. Choose terms τ_i representing b_i. Condition (ii) makes $K_A(\boldsymbol{\tau})$ finitely generated, and Theorem 3.1 makes B finitely presented. Hence S is right coherent.

3.1 Finite equation test

To extract a presentation, one seeks a finite set $\Omega(\boldsymbol{\tau}) \subseteq F_m \times F_m$ satisfying

$$\Omega(\boldsymbol{\tau}) \subseteq K_A(\boldsymbol{\tau}) \quad \text{and} \quad K_A(\boldsymbol{\tau}) \subseteq \langle \Omega(\boldsymbol{\tau}) \rangle.$$

The first inclusion is soundness: every proposed relation is valid in A. The second is completeness: every equality among generated elements follows from the proposed finite relations. Normal forms, critical pairs, finite automata, or direct right-congruence calculations may be used for this step.

4. PURE SUB-ACTS AND FINITE OBSTRUCTIONS

For a finite system $\Sigma(\mathbf{x}, B)$, Definition 2.3 says exactly that solvability of Σ in A implies solvability of the same system in B (Yang & Gould, 2023).

Theorem 4.1 (Purity calculus). Let S be a monoid.

(a) If $C \le B \le A$, C is pure in B, and B is pure in A, then C is pure in A.

(b) Every retract sub-act is pure.

(c) If B_i is pure in A_i for every $i \in I$, then $\prod_{i \in I} B_i$ is pure in $\prod_{i \in I} A_i$.

(d) If $\{B_\lambda : \lambda \in \Lambda\}$ is directed, each B_λ is pure in A, and $B = \bigcup_\lambda B_\lambda$, then B is pure in A.

Proof. Part (a) follows by transferring a solution first from A to B and then from B to C. For (b), apply the retraction coordinatewise to a solution; it preserves the action and fixes the constants. For (c), solve the system in each coordinate and combine the coordinate solutions. For (d), only finitely many constants occur, so directedness places all of them in one B_{λ_0}; purity of B_{λ_0} gives a solution in B.

A **finite purity obstruction** for $B \leq A$ is a finite system over B that is solvable in A but not in B. Thus B is pure in A exactly when no such obstruction exists. The equation $xt = b$ in Example 2.2 is an explicit obstruction.

Proposition 4.2 (One-variable kernel test). Let $B = \langle b_1, \dots, b_m \rangle_S \leq A$. The system

$$xs_i = b_{\alpha_i} t_i \qquad (1 \leq i \leq r)$$

has a solution in B if and only if there are $k \in \{1, \dots, m\}$ and $u \in S$ such that

$$\big((k, us_i), (\alpha_i, t_i)\big) \in K_A(\mathbf{b}) \qquad (1 \leq i \leq r).$$

Proof. If $x = b_k u$, then the equations are equivalent to $b_k u s_i = b_{\alpha_i} t_i$, which is exactly the displayed kernel membership. The converse follows by reading the same equivalence in reverse.

5. COHERENCY CONSEQUENCES

Theorem 5.1. Every finite monoid is right coherent.

Proof. If S and X are finite, then $F_X = X \times S$ is finite, so every finitely presented right S-act is finite. Each finite sub-act B is finitely presented: take one generator x_b for each $b \in B$ and the finite relations $x_b s = x_{bs}$ for $b \in B$ and $s \in S$. Therefore, every finitely generated sub-act of every finitely presented act is finitely presented.

Proposition 5.2. If S is right coherent, then S is weakly right coherent.

Proof. The right regular act $S_S \cong \langle e \mid \varnothing \rangle_S$ is finitely presented. Every finitely generated right ideal is a finitely generated sub-act of S_S, hence is finitely presented.

The converse fails in general: weak right coherency only tests right ideals, whereas right coherency tests generated sub-acts in all finitely presented acts (Brookes et al., 2025). Weak right coherency is controlled by right ideal Howson behavior and finitely generated annihilator congruences (Carson & Gould, 2021; Dasar et al., 2025).

Proposition 5.3. For $s \in S$, define

$$\rho_s = \{(u, v) \in S \times S : su = sv\}.$$

Then $sS \cong S/\rho_s$ as right S-acts. If S is right coherent, then ρ_s is finitely generated.

Proof. The map $\theta_s : S_S \to sS$, $\theta_s(u) = su$, is surjective and has kernel ρ_s, so the quotient theorem gives $sS \cong S/\rho_s$. Under right coherency, the cyclic sub-act $sS \leq S_S$ is finitely presented; Theorem 3.1 then makes its kernel ρ_s finitely generated.

6. REWRITING-COMPATIBLE PRESENTATION EXTRACTION

Assume that $A = \langle X \mid R \rangle_S$ has a rewriting procedure with unique normal forms. Write $p \Downarrow q$ when the normal forms of $p, q \in F_X$ coincide. Then

$$K_A(\boldsymbol{\tau}) = \{((i, u), (j, v)) : \tau_i u \Downarrow \tau_j v\}.$$

A finite set $\Omega \subseteq F_m \times F_m$ is a **finite transition basis** for $\boldsymbol{\tau}$ if $K_A(\boldsymbol{\tau}) = \langle \Omega \rangle$.

Theorem 6.1 (Rewriting extraction criterion). Let $B = \langle [\tau_1]_A, \dots, [\tau_m]_A \rangle_S$. If rewriting produces a finite transition basis Ω, then

$$B \cong \langle e_1, \dots, e_m \mid \Omega \rangle_S.$$

Proof. The equality $K_A(\boldsymbol{\tau}) = \langle \Omega \rangle$ and Theorem 3.1 give the stated presentation.

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A convergent system decides equality, but finite kernel generation still requires finite derivation control. This is the substantive completeness step in rewriting-based coherency arguments (Squier, 1987; Meha, 2023).

7. NUMERICAL MAINTENANCE-STATE MODEL

Consider a weekly record for 12 industrial pumps. Each pump is classified as healthy (H), warning (W), or failed (F). Let

$$S = \{1, m\}, \qquad m^2 = m,$$

where 1 means "no intervention" and m means "complete maintenance and reset." Define the right action on $A = \{H, W, F\}$ by

$$Hm = H, \qquad Wm = H, \qquad Fm = H.$$

Because applying the reset twice has the same effect as applying it once, this is a right S-act. The following fully specified data are used for the calculation.

Table 1. Weekly pump-state record and risk calculation

State	Pumps	After m	Risk per pump	Total risk
H	5	H	0	0
W	4	H	1	4
F	3	H	3	9
Total	**12**	**12 healthy**		**13**

The pre-maintenance total and mean risk are

$$R_0 = 5(0) + 4(1) + 3(3) = 13, \qquad \overline{R}_0 = \frac{13}{12} \approx 1.083.$$

After applying m, all pumps are in state H, so $R_1 = 0$. The action changes the states of $4 + 3 = 7$ pumps, which is

$$\frac{7}{12} \times 100\% \approx 58.3\%.$$

Now take the generating tuple $\mathbf{b} = (W, F)$. Its generated sub-act is

$$B = \langle W, F \rangle_S = \{W, F, H\} = A.$$

The free act F_2 has four elements, and the canonical map satisfies

$$\pi_{\mathbf{b}}(1,1) = W, \quad \pi_{\mathbf{b}}(1, m) = H, \quad \pi_{\mathbf{b}}(2,1) = F, \quad \pi_{\mathbf{b}}(2, m) = H.$$

So the only elements of F_2 which have the same image are (1, m) and (2, m). Therefore

$$K_A(\mathbf{b}) = \langle \big((1, m), (2, m) \big) \rangle,$$

and Theorem 3.1 yields the explicit finite presentation

$$B \cong \langle e_1, e_2 \mid e_1 m = e_2 m \rangle_S.$$

Warning and failed pumps are in the same healthy state after having been serviced for full operation. Lastly, {H} is a retract of A through the constant map to H and, hence, by Theorem 4.1(b), pure. Thus, with numerical calculations, the model possesses the kernel criterion, presentation extraction, and also a purity conclusion in one finite system.

8. CONCLUSION

The finite generating tuple is transformed into an explicit presentation problem by the canonical kernel $K_A(b)$. It is finite and, in particular, a finitely presentable monoid, and is uniformly equivalent with the class of right coherent monoids. Defining relations are provided by finite transition bases, and non-purity by finite equation obstructions. The maintenance model verifies the procedure and outputs a one-relation presentation, along with any required state and risk calculations.

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