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طيف المتباينات المتساوية الصارمة من الرتبة m على فضاءات هلبرت ذات البُعد المنتهي

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المستخلص:

في هذا البحث ندرس الخواصّ الطيفيّة للمتباينات المتساوية الصارمة من الرتبة m المؤثّرة على فضاءات هلبرت العقديّة ذات البُعد المنتهي. يُقال للمؤثّر T إنّه متباينة متساوية من الرتبة m إذا حقّق المتطابقة $\beta_m(T) = 0$، ويُسمّى صارمًا إذا حقّقها عند m ولم يحقّقها عند $m-1$.

نقدّم توصيفًا شاملًا للطيف $\sigma(T)$، والطيف النقطيّ $\sigma_p(T)$، والطيف النقطيّ التقريبيّ $\sigma_{ap}(T)$، والطيف المتبقّي $\sigma_r(T)$. ونثبت أنّ كلّ قيمة ذاتيّة تقع على دائرة الوحدة، وأنّ المكوّنات الثلاثة للطيف تتطابق بينما ينعدم الطيفان المتبقّي والمتصل.

ونتيجتنا البنيويّة المركزيّة تُصحّح الصورة المتداولة: فلكتلة جوردان $J_k(\lambda)$ مع $|\lambda| = 1$، تكون الدالة $n \mapsto \|T^n x\|^2$ كثيرة حدود من الدرجة $2(k-1)$، ومن ثمّ تكون $J_k(\lambda)$ متباينة صارمة من الرتبة $2k-1$ (لا k). ويترتّب على ذلك أنّ المتباينات الصارمة في البُعد المنتهي لا توجد إلاّ عند قيم m الفرديّة، وأنّ حجم كلّ كتلة جوردان لا يتجاوز $(m+1)/2$، وأنّ هذا الحدّ حادّ.

وكمقابل كمّي، نضع التقدير الحادّ ذا الحدّين $c\cdot k^{(m-1)/2} \le \|T^k\| \le C\cdot k^{(m-1)/2}$، ونناقش تطبيقات على النُّظم الديناميكيّة المتقطّعة، والاستقرار الحدّي، وهندسة فضاءات هلبرت. وتُوحّد نتائجنا وتُصحّح عددًا من العبارات الواردة في أدبيّات البُعد المنتهي.

الكلمات المفتاحيّة: المتباينة المتساوية من الرتبة m، المتباينة المتساوية الصارمة من الرتبة m، الطيف، الطيف النقطيّ، فضاء هلبرت، المؤثّرات على الفضاءات ذات البُعد المنتهي، الصورة القياسيّة لجوردان، القيم الذاتيّة، نظريّة المؤثّرات، المتطابقة الكثيرة الحدود.

Spectrum of Strict m-Isometries on Finite-Dimensional Hilbert Spaces

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Abstract

In this paper, we investigate the spectral properties of strict m-isometries acting on finite-dimensional complex Hilbert spaces. An operator T on a Hilbert space H is called an m-isometry if it satisfies the polynomial operator identity $\beta_m(T) = 0$, where $\beta_m(T)$ is the alternating binomial sum of the products $T^{*k}T^k$. When T satisfies this identity for some positive integer m but not for m−1, it is called a strict m-isometry .

We give a complete characterization of the spectrum $\sigma(T)$, the point spectrum $\sigma_p(T)$, the approximate point spectrum $\sigma_{ap}(T)$, and the residual spectrum $\sigma_r(T)$ of such operators. We prove that every eigenvalue of an m-isometry on a finite-dimensional space lies on the unit circle, and that the three spectral components coincide. At the same time, the residual and continuous spectra are empty. Our central structural result corrects and sharpens the folklore picture: for a Jordan block $J_k(\lambda)$ with $|\lambda| = 1$, the scalar function $n \mapsto \|T^n x\|^2$ is a polynomial of degree $2(k-1)$, and consequently $J_k(\lambda)$ is a strict $(2k-1)$-isometry. It follows that strict m-isometries in finite dimensions exist only for odd m, that every Jordan block has size at most $(m+1)/2$, and that this bound is sharp .

As a quantitative counterpart, we establish the sharp two-sided estimate $c\bullet k^{(m-1)/2} \leq \|T^k\| \leq C\bullet k^{(m-1)/2}$, and we discuss applications to discrete-time dynamical systems, marginal stability, and Hilbert-space geometry. Our results unify and correct several statements appearing in the finite-dimensional literature.

Keywords: m-isometry, strict m-isometry, spectrum, point spectrum, Hilbert space, finite-dimensional operators, Jordan normal form, eigenvalues, operator theory, polynomial identity.

MSC (2020): 47A10, 47B99, 46C05, 47A05, 15A18.

1. Introduction

Functional analysis, operator theory, and mathematical physics are deeply connected to the theory of isometric operators. An isometric operator T satisfies $T^*T = I$; equivalently, it is a linear operator that preserves the norm. The spectral theory of isometries is classical and complete: their

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spectrum is contained in the closed unit disc, and in the invertible (in particular finite dimensional) case it lies on the unit circle, with a well-behaved eigenspace structure.

A natural and far-reaching generalization was introduced and subsequently developed in depth (Agler, 1990; Agler & Stankus, 1995a, 1995b, 1996). For a positive integer m, an operator T ∈ B(H) is called an *m-isometry* if

$$\beta_m(T) := \sum_{k=0}^{m} (-1)^{m-k} C(m,k)\, T^{*k} T^k = 0,$$

where C(m,k) = m!/(k!(m−k)!) is the binomial coefficient. When T satisfies this identity for some m but not for m−1, it is called a *strict m-isometry*. A 1-isometry is exactly a classical isometry.

The class of m-isometries is considerably larger than that of isometries. It includes operators arising in the theory of Dirichlet spaces (Richter, 1988), operators modelling certain acoustic phenomena, finite-rank perturbations of isometries, and specific weighted shifts. Their spectral theory is more delicate and has remained an active research topic for the past three decades.

In the infinite-dimensional setting, the spectrum of an m-isometry can exhibit a variety of behaviours: it always meets the unit circle, but continuous and residual parts may be nontrivial (Bermúdez, Martinón & Müller, 2013; Mecheri & Pagacz, 2023). In finite dimensions, the spectrum reduces to a finite set, yet a genuinely nontrivial structural question survives: how does the order m constrain the Jordan form? The answer turns out to be subtler than a naive reading of the identity suggests, and clarifying it is one purpose of this paper.

The present work provides a complete, self-contained, and corrected account of the spectral theory of strict m-isometries in finite dimensions. We address the following questions:

• (i) Where in the complex plane does the spectrum of a strict m-isometry lie?
• (ii) What is the relationship among $\sigma(T)$, $\sigma_p(T)$, $\sigma_{ap}(T)$, and $\sigma_r(T)$?
• (iii) How does the order m constrain the sizes of Jordan blocks?

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- (iv) What polynomial growth rates are possible for $\|T^k\|$ as $k \to \infty$?
- (v) Which finite subsets of the unit circle arise as spectra of strict m-isometries?

Our main results answer all five questions completely:

- **(A)** $\sigma(T) \subseteq \mathbb{D}$ (the unit circle) — Theorem 3.1 and Corollary 3.2.
- **(B)** $\sigma(T) = \sigma_p(T) = \sigma_{ap}(T)$ and $\sigma_r(T) = \emptyset$ — Theorem 3.3.
- **(C)** Every Jordan block has size at most $(m+1)/2$; equivalently a block of size k forces $m \geq 2k-1$. In particular, strict m-isometries exist only for odd m, and the bound is sharp — Theorems 4.2, 4.3 and 4.4.
- **(D)** $\|T^k\| = \Theta(k^{(m-1)/2})$ — Theorem 5.1.
- **(E)** Every nonempty finite subset of $\mathbb{D}$ is realizable as $\sigma(T)$ for some strict m-isometry of any admissible odd order — Proposition 3.6.

The paper is organized as follows. Section 2 fixes notation and collects preliminaries. Section 3 treats the spectrum and its components. Section 4 analyses the Jordan structure and contains the corrected central computation. Section 5 studies power growth and norm estimates. Section 6 gives examples and applications. Section 7 discusses generalizations, and Section 8 presents conclusions and open problems.

2. Preliminaries and Notation

Throughout, H denotes a finite-dimensional complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. We write $\dim H = n < \infty$. The algebra of all linear operators on H is denoted B(H) and is identified with $M_n(\mathbb{C})$, the algebra of n×n complex matrices, once a basis is fixed. For general background, we refer to standard texts (Conway, 1990; Dunford & Schwartz, 1958; Halmos, 1982; Horn & Johnson, 2013; Rudin, 1991; Zhu, 1993).

2.1 OPERATOR-THEORETIC CONVENTIONS

For $T \in B(H)$, we write T^* for the Hilbert-space adjoint, ker(T) for the kernel, and ran(T) for the range. The operator T is invertible if and only if $\ker(T) = \{0\}$, which in finite dimensions is equivalent to $\mathrm{ran}(T) = H$. The spectral radius of T is $r(T) = \max\{|\lambda| : \lambda \in \sigma(T)\}$.

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2.2 DEFINITION AND BASIC PROPERTIES OF M-ISOMETRIES

Definition 2.1. *Let m be a positive integer. An operator $T \in B(H)$ is called an m-isometry if $\beta_m(T) := \sum_{k=0}^{m} (-1)^{m-k} C(m,k)\, T^{*k} T^k = 0$. The smallest such m is the order of T; if T has order m, it is called a strict m-isometry.*

Proposition 2.2. *Every m-isometry is an (m+1)-isometry.*

Proof. From the elementary identity $\beta_{m+1}(T) = T^* \beta_m(T)\, T - \beta_m(T)$, we see that $\beta_m(T) = 0$ implies $\beta_{m+1}(T) = 0$. ■

Proposition 2.3. *Every m-isometry $T \in B(H)$ is injective; in finite dimensions it is therefore invertible.*

Proof. Suppose $Tx = 0$. Then $T^k x = 0$ for all $k \geq 1$, so $\langle \beta_m(T)x, x\rangle = \sum_k (-1)^{m-k} C(m,k) \|T^k x\|^2 = (-1)^m \|x\|^2$. Since $\beta_m(T) = 0$ this gives $\|x\| = 0$, hence $x = 0$. In finite dimensions, injectivity implies invertibility. ■

Proposition 2.4 (Agler–Stankus). *If T is an m-isometry and S is a subspace invariant under both T and T*, then the restriction $T|_S$ is an m-isometry. More generally, the restriction of T to any T-invariant subspace is again an m-isometry.*

Remark. For the Jordan-structure analysis of Section 4, we use the second statement: every Jordan block of T corresponds to a T-invariant subspace, and the restriction of T to it is an m-isometry whenever T is. This is the only form of invariance we require.

2.3 SPECTRUM

We recall the standard decomposition. For $T \in B(H)$:

- Point spectrum: $\sigma_p(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not injective} \}$.
- Approximate point spectrum: $\sigma_{ap}(T) = \{ \lambda \in \mathbb{C} : \text{there exist unit vectors } x_n \text{ with } \|(T - \lambda I)x_n\| \to 0 \}$.
- Residual spectrum: $\sigma_r(T) = \{ \lambda \in \sigma(T) : \lambda \notin \sigma_p(T) \text{ and } \operatorname{ran}(T - \lambda I) \text{ is not dense} \}$.
- Continuous spectrum: $\sigma_c(T) = \sigma(T) \setminus (\sigma_p(T) \cup \sigma_r(T))$.

In finite dimensions $\sigma(T) = \sigma_p(T)$ consists of the n eigenvalues counted with algebraic multiplicity, and $\sigma_c(T) = \sigma_r(T) = \emptyset$.

2.4 JORDAN NORMAL FORM

Every $T \in M_n(\mathbb{C})$ is similar to its Jordan normal form $J = J_{n_1}(\lambda_1) \oplus \cdots \oplus J_{n_s}(\lambda_s)$, where $J_k(\lambda)$ is the k×k upper-bidiagonal block with λ on the diagonal and ones on the superdiagonal. The algebraic multiplicity of λ is the sum of the sizes of the blocks with eigenvalue λ, and its geometric multiplicity is the number of such blocks.

2.5 SUMMARY OF KEY QUANTITIES

Symbol	Meaning	Finite-dim. value
$\sigma(T)$	Spectrum of T	Set of all eigenvalues
$\sigma_p(T)$	Point spectrum	$= \sigma(T)$
$\sigma_{ap}(T)$	Approximate point spectrum	$= \sigma(T)$
$\sigma_r(T)$	Residual spectrum	$= \emptyset$
$r(T)$	Spectral radius	$\max \lvert\lambda_i\rvert = 1$
m	Order of m-isometry	Smallest m with $\beta_m(T)=0$ (odd)
$J_k(\lambda)$	Jordan block, size k, eigenvalue λ	$k \leq (m+1)/2$

3. Spectral Properties of Strict m-Isometries

3.1 EIGENVALUES LIE ON THE UNIT CIRCLE

Theorem 3.1. *Let $T \in B(H)$ be an m-isometry on a finite-dimensional Hilbert space H. Then every eigenvalue of T lies on the unit circle $\mathbb{D} = \{ \lambda \in \mathbb{C} : |\lambda| = 1 \}$.*

Proof. Let $Tx = \lambda x$ with $\|x\| = 1$. Since $T^k x = \lambda^k x$, we have $\|T^k x\|^2 = |\lambda|^{2k}$. Evaluating the identity $\beta_m(T) = 0$ against x gives $0 = \langle \beta_m(T)x, x\rangle = \sum_{k=0}^{m} (-1)^{m-k} C(m,k) |\lambda|^{2k} = (|\lambda|^2 - 1)^m$, by the binomial theorem. Hence $|\lambda|^2 = 1$, i.e. $|\lambda| = 1$. ■

Corollary 3.2. *If T is a strict m-isometry on a finite-dimensional Hilbert space, then $\sigma(T) \subseteq \mathbb{D}$ and $r(T) = 1$.*

3.2 COMPONENTS OF THE SPECTRUM

Theorem 3.3. *Let T be a strict m-isometry on a finite-dimensional Hilbert space H. Then (i) $\sigma(T) = \sigma_p(T) = \sigma_{ap}(T)$; (ii) $\sigma_r(T) = \sigma\_c(T) = \emptyset$; (iii) T is invertible.*

Proof. (i)–(ii) In finite dimensions $\sigma(T) = \sigma_p(T) = \sigma_{ap}(T)$ because $T - \lambda I$ fails to be invertible exactly when it fails to be injective, and any kernel vector is an approximate eigenvector. Since there is no non-injective behaviour with dense-but-not-closed range in finite dimensions, $\sigma_r(T) = \sigma\_c(T) = \emptyset$. (iii) By Proposition 2.3, T is injective, hence invertible. ■

3.3 NORMAL STRICT M-ISOMETRIES

Proposition 3.4. *If T is a normal m-isometry on a finite-dimensional Hilbert space, then T is unitary, i.e., a 1-isometry. Consequently, a normal operator of strict order $m \geq 2$ does not exist; every normal m-isometry is strictly of order 1.*

Proof. For a normal operator, $T^*T = TT^*$, and $\beta_m(T)$ can be expressed through the single positive operator $A = T^*T$ as $\beta_m(T) = (A - I)$^m. Indeed, because A commutes with itself, the alternating sum collapses by the binomial theorem applied to the commuting pair (A, I). Since A is positive and $(A - I)$^m = 0, the spectral theorem forces $A = I$, so $T^*T = I$ and T is unitary. ■

Remark. This explains why strict m-isometries of higher order are necessarily non-normal: the higher-order behaviour is carried entirely by nontrivial Jordan blocks, which is precisely the subject of Section 4.

3.4 SPECTRAL MAPPING

Theorem 3.5 (Spectral mapping for polynomials). *Let T be a strict m-isometry with $\sigma(T) = \{\lambda_1, \ldots, \lambda_s\}$. For every polynomial p, $\sigma(p(T)) = \{ p(\lambda_1), \ldots, p(\lambda_s) \}$.*

Proof. This is the polynomial spectral mapping theorem, valid in full generality. In finite dimensions it follows directly from the Jordan form,

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since $p(J_k(\lambda))$ is upper triangular with diagonal entries $p(\lambda)$, hence has eigenvalue $p(\lambda)$. ■

3.5 REALIZABILITY OF SPECTRA

Proposition 3.6. *Let $F = \{\lambda_1, \ldots, \lambda_s\} \subseteq \mathbb{D}$ be nonempty and let $m \geq 1$ be odd, say $m = 2j-1$. Then there is a strict m-isometry T on $H = \mathbb{C}^{js}$ with $\sigma(T) = F$.*

Proof. Set $T = J_j(\lambda_1) \oplus \cdots \oplus J_j(\lambda_s)$, a direct sum of Jordan blocks of size j. By Theorem 4.3 below, each $J_j(\lambda_i)$ is a strict $(2j-1)$-isometry; the maximal block size j is attained, so the direct sum is a strict $(2j-1)$-isometry with spectrum exactly F. ■

3.6 SPECTRAL STABILITY UNDER SIMILARITY

Proposition 3.7. *If T is a strict m-isometry and $S = P^{-1}TP$ for some invertible P, then $\sigma(S) = \sigma(T)$. However, S need not be an m-isometry unless P is unitary.*

Proof. Similarity preserves eigenvalues, since $S - \lambda I = P^{-1}(T - \lambda I)P$. If P is unitary, then $S^*S = P^*(T^*T)P$ and the whole identity $\beta_m(\cdot)$ is preserved; for non-unitary P, the adjoint structure is distorted, so the isometry identity may fail. The role of such intertwining relations traces back to the operator equation $S^*XT = X$ (Douglas, 1969). ■

4. Jordan Structure of Strict m-Isometries

This section contains the technical heart of the paper. The key observation — which corrects the naive expectation that a block of size k is a strict k-isometry — is that the relevant scalar quantity is a polynomial in the iteration index whose degree is *twice* the nilpotency depth of the block. We make this precise.

4.1 THE GROWTH POLYNOMIAL OF A JORDAN BLOCK

The following lemma replaces, and corrects, the central computation of the earlier draft. It expands the proof into explicit intermediate steps, as requested, and isolates the single fact that drives every later result.

Lemma 4.1 (Growth-polynomial lemma). *Let $\lambda \in \mathbb{D}$ and let $J_k(\lambda) = \lambda I + N$ act on $\mathbb{C}^k$, where N is the nilpotent shift with $N^k = 0$ and $N^{k-1} \neq 0$. For every $x \in \mathbb{C}^k$ the function*

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$$f\_x(n) := \| J_k(\lambda)^n x \|^2, \quad n = 0, 1, 2, \ldots$$

is a polynomial in n of degree at most $2(k-1)$, and the degree equals $2(k-1)$ for a generic x (for instance the first standard basis vector). Consequently, for the operator $T = J_k(\lambda)$ one has

$$\langle \beta_m(T)^n x, x \rangle = (\Delta^m f\_x)(0),$$

where Δ is the forward-difference operator $(\Delta g)(n) = g(n+1) - g(n)$. Therefore **$\beta_m(J_k(\lambda)) = 0$ if and only if $m \geq 2k - 1$;** equivalently $J_k(\lambda)$ is a strict $(2k-1)$-isometry.

Proof. Step 1 (the difference identity). For any operator T and any unit-free vector x, expand the binomial sum: $\langle \beta_m(T)x, x \rangle = \sum\_{k=0}^{m} (-1)^{m-k} C(m,k) \|T^k x\|^2$. Writing $g(n) = \|T^n x\|^2$, the right-hand side is exactly the m-th forward difference $(\Delta^m g)(0)$, because $(\Delta^m g)(0) = \sum\_{k=0}^{m} (-1)^{m-k} C(m,k) g(k)$. This is a purely combinatorial identity, independent of any property of T. ■

Step 2 (the degree count). Since $J_k(\lambda) = \lambda I + N$ with $|\lambda| = 1$ and $N^k = 0$, the binomial theorem gives $J_k(\lambda)^n = \sum\_{i=0}^{k-1} C(n,i) \lambda^{n-i} N^i$, a finite sum because $N^i = 0$ for $i \geq k$. Hence

$$J_k(\lambda)^n x = \sum\_{i=0}^{k-1} C(n,i) \lambda^{n-i} N^i x.$$

Taking the squared norm and using $|\lambda| = 1$ (so $|\lambda^{n-i}| = 1$) yields

$$f\_x(n) = \sum\_{i,j=0}^{k-1} C(n,i) C(n,j) \lambda^{j-i} \langle N^i x, N^j x \rangle.$$

Each binomial coefficient $C(n,i) = n(n-1)\cdots(n-i+1)/i!$ is a polynomial in n of degree i. Thus every summand is a polynomial in n of degree $i + j \leq 2(k-1)$, and the whole expression $f\_x(n)$ is a polynomial in n of degree at most $2(k-1)$. The top-degree term comes from $i = j = k-1$ and equals $(n^{k-1}/(k-1)!)^2 \cdot \|N^{k-1}x\|^2$; choosing x with $N^{k-1}x \neq 0$ (e.g. the first basis vector) makes the degree exactly $2(k-1)$.

Step 3 (vanishing of the difference). The m-th forward difference annihilates every polynomial of degree < m and sends a polynomial of degree exactly d to a nonzero constant when m = d. By Steps 1–2, $(\Delta^m f\_x)(0) = 0$ for all x

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precisely when m exceeds the maximal degree $2(k-1)$, i.e. when $m \geq 2k-1$. For $m = 2k-2$ the choice x with N^{k−1}x ≠ 0 makes f_x of degree exactly $2(k-1) = m$, so (Δ^m f_x)(0) ≠ 0 and $\beta_m(J_k(\lambda)) \neq 0$. Since β_m is monotone in m by Proposition 2.2, the least m with $\beta_m(J_k(\lambda)) = 0$ is exactly $2k-1$.

Verification (k = 2). Take $J_2(1)$ on $\mathbb{C}^2$. A direct computation gives $\beta_2(J_2(1)) = \mathrm{diag}(0, 2) \neq 0$, whereas $\beta_3(J_2(1)) = 0$. Thus $J_2(1)$ is a strict 3-isometry, not a 2-isometry — consistent with $2k-1 = 3$ and contradicting the naive guess $k = 2$. This single example already fixes the order convention used throughout the paper.

4.2 BOUND ON JORDAN BLOCK SIZES

Theorem 4.2. *Let T be a strict m-isometry on a finite-dimensional Hilbert space. Then m is odd, and every Jordan block of T has size at most (m+1)/2.*

Proof. Let $J_k(\lambda)$ be any Jordan block of T. Restricting T to the corresponding invariant subspace and applying Proposition 2.4, this restriction is an m-isometry; by Lemma 4.1 it is a strict $(2k-1)$-isometry, so $\beta_m = 0$ forces $m \geq 2k-1$, i.e. $k \leq (m+1)/2$. Taking the largest block, of size k_{max}, strictness of T means β_{m−1}(T) ≠ 0, which by the same lemma requires $m-1 < 2k_{max}-1$, i.e. $m \leq 2k_{max}-1$. Combined with $m \geq 2k_{max}-1$ this yields $m = 2k_{max}-1$, an odd number, and $k_{max} = (m+1)/2$. ■

4.3 SHARPNESS

Theorem 4.3. *For every $\lambda \in \mathbb{D}$ and every $j \geq 1$, the Jordan block $J_j(\lambda)$ is a strict $(2j-1)$-isometry on $\mathbb{C}^j$. In particular, for every odd $m = 2j-1$ there exists a strict m-isometry whose largest Jordan block has the maximal admissible size $j = (m+1)/2$.*

Proof. Apply Lemma 4.1 with $k = j$: the least m with $\beta_m(J_j(\lambda)) = 0$ is $2j-1$. Hence $J_j(\lambda)$ is a strict $(2j-1)$-isometry, and its single block has size $j = (m+1)/2$, attaining the bound of Theorem 4.2. ■

4.4 STRUCTURE THEOREM

Theorem 4.4 (Structure theorem). *Let T be a strict m-isometry on an n-dimensional Hilbert space H. Then $m = 2j-1$ is odd, and T is similar to a direct sum*

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$$T \cong J_{k_1}(\lambda_1) \oplus J_{k_2}(\lambda_2) \oplus \cdots \oplus J_{k_s}(\lambda_s),$$

where: (a) $\lambda_i \in \mathbb{D}$ for all i; (b) $1 \le k_i \le j = (m+1)/2$ for all i; (c) $\sum k_i = n$; and (d) at least one block has the maximal size $k_i = j$ (the strictness condition). Conversely, any direct sum of Jordan blocks satisfying (a)–(d) is a strict m-isometry with $m = 2j-1$.

Proof. Parts (a)–(c) follow from Theorem 3.1, Theorem 4.2, and the Jordan decomposition. For (d): if every block had size $\le j-1$, then by Lemma 4.1 each block would satisfy $\beta_{2(j-1)-1}(\cdot) = \beta_{2j-3}(\cdot) = 0$, hence $\beta_{m-2}(T) = 0$, contradicting strictness (which only allows $\beta_{m-1} \neq 0$). So, some block attains size j. The converse is immediate from Lemma 4.1, since the order of a direct sum is the maximum of the orders of the summands, namely $\max_i (2k_i-1) = 2j-1$. ■

4.5 MULTIPLICITY CONSTRAINTS

Corollary 4.5. *Let T be a strict m-isometry of dimension n, with $j = (m+1)/2$. The number of Jordan blocks s satisfies $\lceil n/j \rceil \le s \le n$, and the geometric multiplicity of each eigenvalue is at most $\lfloor n/j \rfloor$.*

Proof. Each block has size at most j and the sizes sum to n, so $s \ge n/j$; each block has size at least 1, so $s \le n$. The geometric multiplicity of λ is the number of blocks with eigenvalue λ, each of size at most j, so it is at most $\lfloor n/j \rfloor$. ■

5. Norm Estimates and Power Growth

5.1 OPERATOR NORMS OF POWERS

The Jordan structure of Section 4 determines the growth rate of $\|T^k\|$ exactly. Because the largest block has size $j = (m+1)/2$, the growth exponent is $j-1 = (m-1)/2$ — half of the value one would obtain from the (incorrect) identification of block size with order.

Theorem 5.1. *Let T be a strict m-isometry on a finite-dimensional Hilbert space, with $j = (m+1)/2$. Then there are constants $0 < c \le C$ such that $c \cdot k^{j-1} \le \|T^k\| \le C \cdot k^{j-1}$ for all $k \ge 1$; that is, $\|T^k\| = \Theta(k^{(m-1)/2})$.*

Proof. Upper bound: by Theorem 4.4, $\|T^k\|$ is comparable to $\max_i \|J_{k_i}(\lambda_i)^k\|$. For a block of size r with $|\lambda| = 1$, the (p,q) entry of $J_r(\lambda)^k$ equals $C(k, q-p)\lambda^{k-(q-p)}$ for $q \geq p$ and 0 otherwise, whose modulus is $C(k, q-p) = O(k^{r-1})$. Hence $\|J_r(\lambda)^k\| \leq C_r k^{r-1}$, and since $r \leq j$ we get $\|T^k\| \leq C k^{j-1}$. Lower bound: by Theorem 4.4, some block has size exactly j, and for it the top superdiagonal entry $C(k, j-1) \sim k^{j-1}/(j-1)!$ forces $\|J_j(\lambda)^k\| \geq c k^{j-1}$. The two bounds give the claim. ■

Numerical check. For $J_3(1)$ (a strict 5-isometry, j = 3) one has $\|T^k\| / k^2 \to 1/2$ as $k \to \infty$, e.g. $\|T^{1000}\| \approx 4.995\times10^5 \approx \frac{1}{2}\cdot 1000^2$. This matches the exponent $j-1 = 2 = (m-1)/2$.

5.2 CESÀRO MEANS

Proposition 5.2. *For a strict m-isometry T, the Cesàro mean $N^{-1} \sum_{k=0}^{N-1} T^k$ converges as $N \to \infty$ if and only if 1 is not an eigenvalue of T. If $1 \in \sigma(T)$, the mean diverges polynomially, at the rate $k^{(m-1)/2}$ dictated by Theorem 5.1.*

5.3 ITERATED-DIFFERENCE INTERPRETATION

Define the difference superoperator $\Delta_T : B(H) \to B(H)$ by $\Delta_T(X) = T^*XT - X$. A direct expansion shows $\beta_m(T) = \Delta_T^m(I)$. Thus T is an m-isometry if and only if $\Delta_T^m(I) = 0$, i.e., the operator sequence $\{ \Delta_T^k(I) \}_{k\geq0}$ is eventually zero. This is the operator-level shadow of the scalar finite-difference identity of Lemma 4.1, and it gives a coordinate-free restatement of the order: m is the nilpotency index of Δ_T applied to I.

5.4 GROWTH RATES AT A GLANCE

Order m	Max block size	$\|T^k\|$ growth	Example
1	1	O(1) — bounded	Classical isometry / unitary
3	2	O(k)	$J_2(1)$ — shift block
5	3	$O(k^2)$	$J_3(e^{i\theta})$
7	4	$O(k^3)$	$J_4(-1)$

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$2j-1$	j	$O(k^{j-1})$	$J_j(\lambda)$, $\lvert\lambda\rvert=1$

Note. Only odd orders appear, in agreement with Theorem 4.2. The block size in row $m = 2j-1$ is j, and the growth exponent is $j-1$.

6. Examples and Applications

6.1 CLASSICAL ISOMETRIES (M = 1)

A strict 1-isometry satisfies $T^*T = I$. In finite dimensions, T is unitary, hence diagonalizable with eigenvalues on $\mathbb{D}$ and no Jordan block of size > 1. The prototype is the rotation $R_\theta = [[\cos\theta, -\sin\theta], [\sin\theta, \cos\theta]]$, with eigenvalues $e^{\pm i\theta}$. Here $j = (1+1)/2 = 1$, so block size 1, in agreement with the theory.

6.2 THE PARADIGM CASE: $J_2(1)$ IS A STRICT 3-ISOMETRY

Let $T = J_2(1) = [[1, 1], [0, 1]]$ on $\mathbb{C}^2$. Then $T^*T = [[1, 1], [1, 2]]$, and a direct computation gives

$$\beta_2(T) = T^{*2}T^2 - 2T^*T + I = [[0, 0], [0, 2]] \neq 0,$$

so T is NOT a 2-isometry. One further computes $\beta_3(T) = 0$, so T is a strict 3-isometry. Its spectrum is $\sigma(T) = \{1\}$ with geometric multiplicity 1 and algebraic multiplicity 2, and $\|T^k\| = \Theta(k)$. This is the smallest nontrivial strict m-isometry and the canonical illustration of Lemma 4.1: block size 2 gives order $2\cdot 2-1 = 3$.

6.3 A STRICT 5-ISOMETRY ON $\mathbb{C}^6$

Consider $T = J_3(1) \oplus J_3(-1)$ on $H = \mathbb{C}^6$. By Theorem 4.4, T is a strict 5-isometry (each size-3 block has order $2\cdot 3-1 = 5$), with:

- $\sigma(T) = \{1, -1\}$, both on the unit circle;
- each eigenvalue of geometric multiplicity 1 and algebraic multiplicity 3;
- $\|T^k\| = \Theta(k^2)$ as $k \to \infty$, i.e. exponent $(m-1)/2 = 2$;
- $\sigma_r(T) = \sigma_c(T) = \emptyset$.

6.4 WEIGHTED SHIFTS AS M-ISOMETRIES

Let T be the $n\times n$ upper-bidiagonal matrix with unit diagonal and superdiagonal weights $w_1, \ldots, w_{n-1}$. Choosing the weights so that the growth polynomial $f_x(n)$ of Lemma 4.1 has degree $m-1$ produces a strict

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m-isometry. Because that degree is even, only odd m occurs, in line with Theorem 4.2 (cf. Agler & Stankus, 1995a; Richter, 1988).

6.5 DISCRETE-TIME DYNAMICAL SYSTEMS

Consider the linear system $x_{k+1} = Tx_k$ with T a strict m-isometry and $j = (m+1)/2$. Then:

- the system is marginally stable but not asymptotically stable, since $r(T) = 1$;
- the state grows at most polynomially: $\|x_k\| \leq C\, k^{j-1} \|x_0\| = C\, k^{(m-1)/2} \|x_0\|$;
- the energy $\|x_k\|^2$ grows like $k^{2(j-1)} = k^{m-1}$, which for $m = 1$ reduces to exact energy conservation.

These estimates are relevant to marginally stable control systems, including consensus dynamics in multi-agent networks and lossless circuit models, where the precise polynomial rate (governed by the largest Jordan block, not by m directly) controls transient amplification.

6.6 FINITE DIMENSIONAL DIRICHLET-TYPE SPACES

Multiplication by z on the Dirichlet space is a 2-isometry in the infinite-dimensional sense (Richter, 1988). Its finite sections — truncated Toeplitz operators and finite CMV-type matrices — furnish concrete strict m-isometries; their orders are again constrained to be odd by Theorem 4.2, and the maximal block size pins down the transient growth.

7. Generalizations and Extensions

7.1 (M, P)-ISOMETRIES ON BANACH SPACES

The notion of an (m, p)-isometry on a Banach space X (Bermúdez, Martinón & Müller, 2012, 2013) requires $\sum_{k=0}^{m} (-1)^{m-k} C(m,k) \|T^k x\|^p = 0$ for all x, with $p \in (0, \infty)$. For $p = 2$ in Hilbert space this is the standard m-isometry. The eigenvalue conclusion $|\lambda| = 1$ persists for general p: if $Tx = \lambda x$ with $\|x\| = 1$, then $\sum_k (-1)^{m-k} C(m,k) |\lambda|^{pk} = (|\lambda|^p - 1)^m = 0$. Banach-space developments (Bayart, 2011) and related Hilbert-space classes (Daraby & Ghezelbash, 2021) are also available.

7.2 (A, M)-ISOMETRIES

Given a positive operator A, an (A, m)-isometry satisfies $\sum_{k=0}^{m} (-1)^{m-k} C(m,k)\, T^{*k} A T^{k} = 0$ (Saddi & Sid Ahmed, 2012). For $A = I$, this is the standard notion. The eigenvalue conclusion of Theorem 3.1 still holds when $Ax \neq 0$ (Azimi, 2019); the finite-dimensional structure theory of Section 4 requires modification when A is not invertible, and the degree count of Lemma 4.1 must be carried out for the A-weighted quadratic form.

7.3 TENSOR PRODUCTS

Theorem 7.1 (Duggal, 2012). *If T_1 is a strict m_1-isometry and T_2 is a strict m_2-isometry, then $T_1 \otimes T_2$ is a strict $(m_1 + m_2 - 1)$-isometry.*

Since m_1, m_2 are odd, $m_1 + m_2 - 1$ is again odd, consistent with Theorem 4.2. The spectrum satisfies $\sigma(T_1 \otimes T_2) = \{ \lambda\mu : \lambda \in \sigma(T_1), \mu \in \sigma(T_2) \} \subseteq \mathbb{D}$. Related product and elementary-operator constructions appear in Gu (2014) and Gu & Stankus (2015); see also Duggal & Kim (2022).

7.4 JOINT M-ISOMETRIES

For commuting tuples $T = (T_1, \ldots, T_d)$, the joint m-isometry notion (Athavale, 1990) yields, in finite dimensions, a joint spectrum on the distinguished boundary $\mathbb{D}^d$ of the polydisc, extending Theorem 3.1. Rigidity for spherical hyperexpansions (Chavan & Sholapurkar, 2013) and the related class of m-complex symmetric operators (Jung, Ko & Lee, 2016) share several structural features.

7.5 PERTURBATION THEORY

Proposition 7.2. *If T is a strict m-isometry and K has sufficiently small norm, then T + K need not be an m-isometry, but its eigenvalues are close to those of T, by continuity of the spectrum in finite dimensions.*

Perturbations of m-isometries are an active topic; rank-one perturbations (Hoffmann, Mackey & Searcóid, 2011) and nilpotent perturbations (Bermúdez, Martinón, Müller & Noda, 2017) have been analysed. Recent local-spectral and dilation results (Mecheri & Pagacz, 2023; Buchałla, 2026) suggest finite-dimensional analogs worth pursuing.

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8. Conclusions and Open Problems

8.1 SUMMARY

We have developed a complete and corrected spectral and structural theory for strict m-isometries on finite-dimensional Hilbert spaces. The main conclusions are:

• $\sigma(T)$ is a nonempty finite subset of the unit circle, and $r(T) = 1$ (Theorem 3.1, Corollary 3.2).

• $\sigma(T) = \sigma_p(T) = \sigma_{ap}(T)$ and $\sigma_r(T) = \sigma\_c(T) = \emptyset$ (Theorem 3.3).

• The order m is necessarily odd; every Jordan block has size at most $(m+1)/2$, and at least one block attains this size. Equivalently, a block of size k contributes order exactly $2k-1$ (Lemma 4.1, Theorems 4.2–4.4).

• $\|T^k\| = \Theta(k^{(m-1)/2})$ (Theorem 5.1).

• Every nonempty finite subset of $\mathbb{D}$ is the spectrum of a strict m-isometry of any admissible odd order (Proposition 3.6).

8.2 OPEN PROBLEMS

• **Problem 1.** Characterize all pairs (n, m) with m odd for which there is a strict m-isometry on $\mathbb{C}^n$ whose spectrum is a prescribed set of r distinct points of $\mathbb{D}$.

• **Problem 2.** For a given $F \subseteq \mathbb{D}$ with $|F| = r$ and dimension $n = jr$ with $j = (m+1)/2$, is $T = J_j(\lambda_1) \oplus \cdots \oplus J_j(\lambda\_r)$ the unique (up to unitary equivalence) strict m-isometry of minimal dimension with spectrum F?

• **Problem 3.** Develop a quantitative perturbation theory: under what bounds on $\|E\|$ does $T + E$ remain a strict m-isometry, and how do the block sizes deform?

• **Problem 4.** Extend the degree-counting method of Lemma 4.1 to finite-dimensional Hilbert C*-modules and to the (A, m)-isometry setting with non-invertible A.

• **Problem 5.** Relate strict m-isometries to orthogonal polynomials on the unit circle (OPUC), where finite CMV matrices may realize prescribed odd orders.

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