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معايير فعلية للتكافؤ الموريتا المتدرّج لجبر مسارات ليفيت المرتبط بمخططات أفعال أنصاف الزمر

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المستخلص:

ليكن (S) مونويدًا، و(A) فعلاً أيسر على (S)، ولتكن (C) مجموعة جزئية منتهية غير خالية من (S). ندرس مخطط فعل نصف الزمرة المستحث بالمجموعة (C)، ويرمز إليه بـ $\Gamma_C(A)$، وتتكون مجموعة رؤوسه من عناصر (A)، في حين تسجل حوافه الموجهة فعل عناصر (C) مع الاحتفاظ بالتعددية. ونحوّل مفهومي المجموعات الوراثية والمشبعة في نظرية المخططات إلى مفهومين داخليين في نظرية الأفعال، نسميهما المجموعات (C)−الوراثية و(C)−المشبعة، ونستخدم هذا التقابل لوصف المثاليات المتدرجة في $L_K(\Gamma_C(A))$. وعندما يكون الفعل منتهيًا، نحدد المونويد الموهوب ونستنتج عرضًا لـ $K_0^{\mathrm{gr}}\left(L_K(\Gamma_C(A))\right)$ مباشرة من مصفوفة الفعل (M_C) وتتمثل نتيجتنا الرئيسة في تقديم شرط كافٍ، مصاغ بلغة الأفعال، للتكافؤ الموريتا المتدرج، نسميه التكافؤ بالإزاحة المتراصفة المستحث بالفعل .ويصاغ هذا الشرط باستعمال تقابلات جسرية منتهية بين الأفعال، ويؤدي إلى تكافؤ إزاحة متراصف بين مصفوفات المجاورة المرتبطة .كما نثبت معيارًا ثانيًا يعتمد على الأفعال الجزئية الكاملة والزوايا المتجانسة الكاملة، ويوفر إجراء اختزال فعالًا .وبالنسبة إلى الأفعال المونوجينية المنتهية، نبرهن أن جبر مسارات ليفيت لمخطط الفعل الدالي الكامل متكافئ موريتياً، بالمعنى المتدرج، مع جبر نواة دوراته، في حين تحتفظ $\left(K_0^{\{gr\}}\right)$ ببيانات أطوال الدورات . وتبين الأمثلة، ومنها مثال مضاد لعدم القابلية للرفع، أن الإطار الفعلي المقترح إطار تقييدي حقيقي، لا مجرد إعادة صياغة لنظرية المخططات.

الكلمات المفتاحية :جبر مسارات ليفيت؛ فعل نصف الزمرة؛ مخطط الفعل؛ التكافؤ الموريتا المتدرّج؛ زمرة غروتينديك المتدرّجة؛ المونويد الموهوب؛ مصفوفة الفعل؛ تكافؤ الإزاحة المتراصف؛ الفعل المونوجيني المنتهي.

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Act-Theoretic Criteria for Graded Morita Equivalence of Leavitt Path Algebras Associated with Semigroup-Act Graphs

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Abstract

Let S be a monoid, A a left S-act, and C a finite nonempty subset of S. We study the C-induced semigroup-act graph $\Gamma_C(A)$, whose vertex set is A and whose directed edges record the action of elements of C, with multiplicities retained. We translate the graph-theoretic notions of hereditary and saturated subsets into intrinsic act-theoretic notions, termed C-hereditary and C-saturated subsets, and use this correspondence to describe the graded ideals of $L_K(\Gamma_C(A))$. When the act is finite, we determine the talented monoid and derive a presentation of $K_0^{\mathrm{gr}}\left(L_K(\Gamma_C(A))\right)$ directly from the action matrix M_C. Our main result provides an act-theoretic sufficient condition for graded Morita equivalence, called act-induced aligned shift equivalence. It is formulated in terms of finite bridge correspondences between acts and yields aligned shift equivalence of the associated adjacency matrices. We also establish a second criterion based on full subacts and full homogeneous corners, which gives an effective reduction procedure. For finite monogenic acts, we show that the Leavitt path algebra of the full functional action graph is graded Morita equivalent to the algebra of its cycle core, while K_0^{gr} retains the cycle-length data. The examples, including a non-liftability counterexample, demonstrate that the proposed act-theoretic framework is genuinely restrictive rather than a mere reformulation of graph theory.

Keywords: Leavitt path algebra; semigroup act; action graph; graded Morita equivalence; graded Grothendieck group; talented monoid; action matrix; aligned shift equivalence; finite monogenic act.

1. Introduction

Leavitt path algebras lie at the intersection of directed graph theory, noncommutative ring theory, graded algebra, and K-theoretic classification. They were introduced by Abrams and Aranda Pino (2005), while a parallel construction emerged from the nonstable K-theory of graph algebras in the work of Ara, Moreno and Pardo (2007). Since then, they have become a prominent class of naturally Z-graded algebras. Their structure is controlled by graph-theoretic data, including hereditary and saturated subsets, cycles, and exits; in particular, such data determine their ideal structure (Tomforde, 2007).

A natural way to construct a directed graph from a semigroup is through a left semigroup act. Mukherjee, Mukherjee and Sardar (2023) formalized this construction as the C-induced action graph of a semigroup act and showed that it encompasses Cayley-type graphs, transformation graphs, and related action graphs. In subsequent work, Mukherjee, Mukherjee and Sardar (2024) studied the Leavitt path algebras associated with these action graphs and characterized conditions for primality, primitivity, ideal simplicity, graded ideal simplicity, and purely infinite simplicity.

The present paper addresses a different problem: graded Morita equivalence and graded classification invariants for Leavitt path algebras arising from semigroup-act graphs. The graded Grothendieck group K_0^{gr} has been extensively studied as a central invariant of Leavitt path algebras (Hazrat, 2013; Ara and Pardo, 2014; Vas, 2023), while the talented monoid encodes the positive cone in graph-theoretic terms (Cordeiro, Goncalves and Hazrat, 2022). Recent work on shift equivalence, aligned shift equivalence, and graded Morita equivalence has further strengthened the connection between symbolic dynamics and the classification of Leavitt path algebras (Abrams, Ruiz and Tomforde, 2024; Aguyar Brix, Dor-On, Hazrat and Ruiz, 2025; Cortinas and Hazrat, 2024).

These tools, however, have not yet been formulated directly in the language of semigroup acts. Although graph-theoretic results can be applied to an action graph, our aim is different: we seek criteria expressed in terms of the act data itself, including C-stability, C-saturation, action matrices, bridge correspondences, full subacts, and finite monogenic dynamics. This distinction matters because graphs of the form $\Gamma_C(A)$ are highly structured. In particular, once the finite set C is fixed, every vertex emits exactly |C| edges, counted with multiplicity. Thus, the framework is not simply graph theory under a different name.

We do not claim a complete classification of all Leavitt path algebras arising from semigroup-act graphs. Instead, we establish new sufficient conditions for graded Morita equivalence, provide computable invariants, and obtain a concrete graded Morita reduction for finite monogenic acts. The results are deliberately formulated conservatively so that the scope of each statement remains precise.

1.1 MAIN CONTRIBUTIONS

1. We define and use a multiplicity-preserving C-induced graph $\Gamma_C(A)$ for a finite subset C of a monoid acting on A.
2. We characterize hereditary and saturated subsets of $\Gamma_C(A)$ as C-hereditary and C-saturated subsets of A.
3. We reformulate the graded ideals of $L_K(\Gamma_C(A))$ using only act-theoretic data. This is treated as a translation tool rather than the principal novelty.
4. For finite acts, we compute the talented monoid and K_0^{gr} from the action matrix M_C.
5. We introduce act-induced aligned shift equivalence via finite bridge correspondences and prove that it implies graded Morita equivalence.
6. We prove a full-subact criterion giving graded Morita reductions by full homogeneous corners.
7. For finite monogenic acts, we reduce the graded Morita type to the cycle-core graph and retain cycle-length information in K_0^{gr}.
8. We include examples and a counterexamples showing that graph-theoretic graded Morita equivalences need not remain inside the class of fixed-C semigroup-act graphs.

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2. Preliminaries

2.1 SEMIGROUPS AND ACTS

A semigroup is a set S equipped with an associative binary operation. A monoid is a semigroup with an identity element, denoted by 1. A left S-act is a nonempty set A together with a map $S \times A \to A$, written $(s, a) \mapsto sa$, such that

$$(st)a = s(ta)$$

for all $s, t \in S$ and $a \in A$. If S is a monoid, we also require $1a = a$ for all $a \in A$. We write A for the underlying set of the left S-act.

A subset B of A is a subact if $sb \in B$ for all $s \in S$ and $b \in B$. In this paper we also allow the empty subset when discussing hereditary subsets and ideal lattices. A morphism of S-acts $f: A \to B$ is a map satisfying $f(sa) = sf(a)$. A cyclic act is an act generated by one element. A monogenic monoid is a monoid generated by one element, usually written $\langle t \rangle$.

2.2 DIRECTED GRAPHS

A directed graph is a quadruple

$$E = (E^0, E^1, s, r),$$

where E^0 is the vertex set, E^1 is the edge set, and $s, r: E^1 \to E^0$ are the source and range maps. A path of length n is a sequence $e_1 \cdots e_n$ with $r(e_i) = s(e_{i+1})$. A cycle is a nontrivial path with the same initial and terminal vertex and no repeated intermediate vertices. A sink is a vertex emitting no edges. A graph is row-finite if every vertex emits finitely many edges.

A subset H of E^0 is hereditary if $s(e) \in H$ implies $r(e) \in H$ for all $e \in E^1$. For a row-finite graph, a hereditary subset H is saturated if every regular vertex v with $r\left(s^{-1}(v)\right) \subseteq H$ also belongs to H.

2.3 LEAVITT PATH ALGEBRAS

Let K be a field and E a row-finite directed graph. The Leavitt path algebra $L_K(E)$ is the K-algebra generated by vertices $\{v: v \in E^0\}$, edges $\{e: e \in E^1\}$

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and ghost edges $\{e^\star : e \in E^1\}$, subject to the standard Cuntz-Krieger relations:

$$vw = \delta_{v,w} v \quad (v, w \in E^0),$$

$$s(e)e = e = er(e), \qquad r(e)e^\star = e^\star = e^\star s(e),$$

$$e^\star f = \delta_{e,f} r(e) \quad (e, f \in E^1),$$

$$v = \sum_{s(e)=v} e\, e^\star \quad \text{for every regular vertex } v.$$

The algebra $L_K(E)$ is naturally $\mathbb{Z}$-graded by

$$\deg(v) = 0, \qquad \deg(e) = 1, \qquad \deg(e^\star) = -1.$$

2.4 GRADED MORITA EQUIVALENCE

Two $\mathbb{Z}$-graded rings R and S are graded Morita equivalent if their categories of graded right modules are equivalent through a degree-preserving equivalence, up to the standard shift functors. A homogeneous idempotent p in a graded ring R is full if $RpR = R$. If p is a full homogeneous idempotent, then R and pRp are graded Morita equivalent. We use this standard graded Morita principle repeatedly in Sections 7 and 8.

2.5 $\boldsymbol{K_0^{\mathrm{gr}}}$ AND TALENTED MONOIDS

For a graded ring R, the group $K_0^{\mathrm{gr}}(R)$ is the Grothendieck group of the category of finitely generated graded projective R-modules. When $R = L_K(E)$, the shift of grading gives $K_0^{\mathrm{gr}}(R)$ the structure of a $\mathbb{Z}[x, x^{-1}]$-module. The talented monoid T_E of a directed graph E records the positive cone of $K_0^{\mathrm{gr}}(L_K(E))$ in a graph-theoretic way and has been developed as a refined invariant for graph algebras (Cordeiro, Goncalves and Hazrat, 2022).

3. $\boldsymbol{C}$-Induced Semigroup-Act Graphs

Let S be a monoid, let A be a left S-act, and let $C \subseteq S$ be a finite nonempty subset. Define the C-induced semigroup-act graph $\Gamma_C(A)$ by

$$\Gamma_C(A) = (E^0, E^1, s, r),$$

$$E^0 = A, \qquad E^1 = C \times A,$$

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$$s(c,a)=a, \qquad r(c,a)=ca.$$

This is a directed multigraph. The multiplicity convention is essential: if $c_1a=c_2a$ with $c_1\neq c_2$, then (c_1,a) and (c_2,a) are distinct edges. Leavitt path algebras depend on edge multiplicity, so collapsing such edges would lose information.

Proposition 3.1 (Row-finiteness). If C is finite, then $\Gamma_C(A)$ is row-finite.

Proof. For each $a\in A$, the edges emitted by a are exactly the pairs (c,a) with $c\in C$. Hence $|s^{-1}(a)|=|C|$, which is finite. Thus, every vertex emits finitely many edges.

Proposition 3.2 (Finiteness). If A and C are finite, then $\Gamma_C(A)$ is a finite directed graph.

Proof. The vertex set is A and the edge set is $C\times A$. Both are finite under the hypotheses.

Proposition 3.3 (Subacts as hereditary subsets). Assume $C=S$. Then the hereditary subsets of $\Gamma_S(A)$ are precisely the S-stable subsets of A, and hence the subacts of A, allowing the empty subset by convention.

Proof. A subset H of A is hereditary in $\Gamma_S(A)$ if and only if for every $a\in H$ and every edge with source a, the range of that edge also belongs to H. The edges with source a are (s,a) for $s\in S$, and their ranges are sa. Therefore H is hereditary if and only if $a\in H$ implies $sa\in H$ for all $s\in S$, which is precisely S-stability.

4. $\boldsymbol{C}$-Hereditary Saturated Subacts and Graded Ideals

We now translate the graph-theoretic conditions governing graded ideals into the language of the act. The results in this section are not presented as the main novelty; they are the dictionary that permits the later Morita-theoretic criteria to be stated act-theoretically.

A subset $H\subseteq A$ is called C-hereditary if

$$a\in H,\ c\in C \quad\Rightarrow\quad ca\in H.$$

Proposition 4.1 (Hereditary subsets). A subset $H\subseteq A$ is hereditary in $\Gamma_C(A)$ if and only if it is C-hereditary.

Proof. By definition, H is hereditary in $\Gamma_C(A)$ exactly when $s(e)\in H$ implies $r(e)\in H$ for every edge e. Since the edges are pairs (c,a), this condition says: if $a \in H$, then $ca \in H$ for every $c \in C$. This is exactly C-heredity.

A C-hereditary subset $H \subseteq A$ is called C-saturated if

$$ca \in H \ \text{ for all } c \in C \quad \Rightarrow \quad a \in H.$$

Proposition 4.2 (Saturated subsets). A C-hereditary subset $H \subseteq A$ is saturated in $\Gamma_C(A)$ if and only if it is C-saturated.

Proof. Since C is finite and nonempty, every vertex a emits exactly $|C|$ edges and hence is a regular vertex. The ranges of all edges emitted by a are precisely the elements ca with $c \in C$. The graph saturation condition therefore says that if all ranges of edges emitted by a lie in H, then a lies in H. This is exactly the stated C-saturation condition.

For $X \subseteq A$, define $\mathrm{Sat}_C(X)$ to be the smallest C-hereditary and C-saturated subset of A containing X. It may be constructed by the iteration

$$X_0 = X,$$

$$X_{n+1} = X_n \cup \{ca: a \in X_n,\ c \in C\} \cup \{a \in A: ca \in X_n \text{ for all } c \in C\}.$$

Then

$$\mathrm{Sat}_C(X) = \bigcup_{n \geq 0} X_n.$$

The proof is the usual closure-operator argument: the sequence is increasing, the union is C-hereditary and C-saturated, and every C-hereditary C-saturated set containing X contains every X_n.

Proposition 4.3 (Graded ideals). Let C be finite and nonempty. For the row-finite graph $\Gamma_C(A)$, the graded ideals of $L_K(\Gamma_C(A))$ correspond to C-hereditary C-saturated subsets of A. The correspondence is

$$H \mapsto I_H = \langle h: h \in H \rangle.$$

Proof. For row-finite Leavitt path algebras, graded ideals are classified by hereditary saturated subsets of the vertex set (Tomforde, 2007; Ara, Moreno and Pardo, 2007). By Propositions 4.1 and 4.2, these are exactly the C-hereditary C-saturated subsets of A. Substituting this translation into the standard graph-theoretic classification yields the stated lattice correspondence.

5. Action Matrices, Talented Monoids and K_0^{gr}

Assume throughout this section that

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$$A = \{a_1, \dots, a_n\}$$

is finite and that $C \subseteq S$ is finite. Define the action matrix $M_C = (m_{ij}) \in M_n(\mathbb{N})$ by

$$m_{ij} = \left|\{c \in C : ca_i = a_j\}\right|.$$

This is exactly the adjacency matrix of $\Gamma_C(A)$, with row i corresponding to source a_i and column j corresponding to range a_j. The use of multiplicities is essential.

Theorem 5.1 (Talented monoid presentation). Let A be a finite S-act and $C \subseteq S$ finite. The talented monoid $T_{\Gamma_C(A)}$ is generated by symbols

$$a_i(k), \qquad 1 \leq i \leq n,\ k \in \mathbb{Z},$$

with defining relations

$$a_i(k) = \sum_{c \in C} (ca_i)\,(k+1).$$

Equivalently,

$$a_i(k) = \sum_{j=1}^{n} m_{ij}\, a_j(k+1).$$

Proof. For a row-finite graph E, the talented monoid T_E is generated by $v(k)$, with $v \in E^0$ and $k \in \mathbb{Z}$, subject to relations

$$v(k) = \sum_{s(e)=v} r\,(e)(k+1).$$

In $E = \Gamma_C(A)$, the edges with source a_i are exactly (c, a_i), $c \in C$, and their ranges are ca_i. Substitution gives the first displayed relation. Grouping terms according to the equality $ca_i = a_j$ gives the equivalent matrix form.

Theorem 5.2 (K_0^{gr} from the action matrix). There is an isomorphism of $\mathbb{Z}[x, x^{-1}]$-modules

$$K_0^{\mathrm{gr}}\left(L_K(\Gamma_C(A))\right) \cong \mathbb{Z}[x, x^{-1}]^n \Big/ \left\langle e_i - x\sum_{j=1}^{n} m_{ij}\, e_j : 1 \leq i \leq n \right\rangle.$$

Proof. The graded Grothendieck group is obtained by group-completing the graded projective-class monoid, with the grading shift represented by multiplication by x. The relation $a_i(k) = \sum_j m_{ij}\, a_j(k+1)$ becomes, at level $k = 0$,

$$e_i = x \sum_j m_{ij}\, e_j.$$

Hence the stated quotient presentation follows. This agrees with the standard K_0^{gr} presentation for finite graph Leavitt path algebras, expressed here directly through the action matrix M_C (Hazrat, 2013; Cordeiro, Goncalves and Hazrat, 2022).

This theorem is computationally useful: one may compute K_0^{gr} from the action table of C on A without first drawing the graph. In examples, it is often enough to write down M_C and reduce the corresponding $\mathbb{Z}[x, x^{-1}]$-module presentation.

6. Act-Induced Aligned Shift Equivalence

This section presents the paper's main sufficient criterion. To keep the criterion at the level of semigroup acts rather than matrices alone, we first introduce finite bridge correspondences between acts. The cardinalities of these correspondences then determine the matrices used in the aligned shift-equivalence argument.

6.1 BRIDGE CORRESPONDENCES

Let (A, C) and (B, D) be finite act systems, with action matrices M_C and M_D. For $\ell \geq 1$, define

$$W_C^\ell(a, a') = \{(c_1, \dots, c_\ell) \in C^\ell : c_\ell \cdots c_1 a = a'\},$$
$$W_D^\ell(b, b') = \{(d_1, \dots, d_\ell) \in D^\ell : d_\ell \cdots d_1 b = b'\}.$$

Thus $\left|W_C^\ell(a_i, a_j)\right| = \left(M_C^\ell\right)_{ij}$ and similarly for D.

A lag-ℓ act bridge from (A, C) to (B, D) consists of finite sets $P(a, b)$ and $Q(b, a)$, for $a \in A$ and $b \in B$, together with the following bijective data:

$$L_P(a, b) \colon \bigsqcup_{c \in C} P(ca, b)\, o \bigsqcup_{\substack{d \in D,\, b' \in B \\ db' = b}} P(a, b'),$$

$$L_Q(b, a) \colon \bigsqcup_{d \in D} Q(db, a)\, o \bigsqcup_{\substack{c \in C,\, a' \in A \\ ca' = a}} Q(b, a'),$$

$$\Omega_A(a, a') \colon \bigsqcup_{b \in B} P(a, b) \times Q(b, a') \to W_C^\ell(a, a'),$$

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$$\Omega_B(b,b'): \bigsqcup_{a\in A} Q(b,a)\times P(a,b') \to W_D^{\ell}(b,b').$$

The first two families of bijections encode one-step compatibility between the C-action on A and the D-action on B. The last two identify a pair of successive bridge correspondences with an l-step word in the relevant original act. We call the bridge aligned when these bijections are compatible with decomposing words at their first and last edges; equivalently, the cardinality matrices defined below form an aligned shift equivalence in the sense of Leavitt path algebras (Abrams, Ruiz and Tomforde, 2024). If the induced shift equivalence also preserves the distinguished order units, we call the bridge unital aligned.

Lemma 6.1 (Cardinality matrices). Let $R=(r_{ab})$ and $S=(s_{ba})$ be defined by

$$r_{ab}=|P(a,b)|, \qquad s_{ba}=|Q(b,a)|.$$

Then a lag-ℓ act bridge gives

$$M_CR=RM_D, \qquad SM_C=M_DS, \qquad RS=M_C^{\ell}, \qquad SR=M_D^{\ell}.$$

Proof. For the first equality, the (a,b)-entry of M_CR counts pairs consisting of $c\in C$ and an element of $P(ca,b)$. The bijection $L_P(a,b)$ identifies this set with the set counted by the (a,b)-entry of RM_D, namely pairs consisting of an element of $P(a,b')$ and $d\in D$ such that $db'=b$. Thus $M_CR=RM_D$. The proof of $SM_C=M_DS$ is the same using L_Q. The (a,a')-entry of RS counts triples b, $p\in P(a,b)$, $q\in Q(b,a')$, and $\Omega_A(a,a')$ identifies these with ℓ-step C-words from a to a'. Hence $RS=M_C^{\ell}$. The proof of $SR=M_D^{\ell}$ uses Ω_B.

Theorem 6.2 (Act-induced aligned shift equivalence). Let (A,C) and (B,D) be finite act systems such that $\Gamma_C(A)$ and $\Gamma_D(B)$ have no sinks. If there exists a unital aligned lag-ℓ act bridge between them, then

$$L_K\big(\Gamma_C(A)\big) \quad \text{and} \quad L_K\big(\Gamma_D(B)\big)$$

are graded Morita equivalent. Moreover, in the unital degree-preserving case their ordered $\mathbb{Z}[x,x^{-1}]$-modules K_0^{gr} are isomorphic with distinguished order unit.

Proof. By Lemma 6.1, the cardinality matrices R and S give a shift equivalence of the adjacency matrices M_C and M_D of lag ℓ. The alignment hypothesis is precisely the additional compatibility needed to grade the bridging bimodule used in the proof of graded Morita equivalence for finite graphs with no sinks. Applying the graded Morita theorem for aligned shift equivalence (Abrams, Ruiz and Tomforde, 2024), and the unital refinement when available (Aguyar Brix, Dor-On, Hazrat and Ruiz, 2025), yields the graded Morita equivalence of $L_K(\Gamma_C(A))$ and $L_K(\Gamma_D(B))$. The statement about K_0^{gr} follows because graded Morita equivalence induces an isomorphism of ordered graded Grothendieck groups, and the unital degree-preserving hypothesis preserves the distinguished order unit.

The act-theoretic content is the restriction that R and S must arise from bridge correspondences P and Q satisfying the four families of bijections above. A general matrix shift equivalence need not be induced by such act data.

This theorem is a sufficient criterion, not a complete classification theorem. Its value is that it gives an act-level route into the modern aligned-shift-equivalence machinery without treating $\Gamma_C(A)$ as an arbitrary graph.

7. Full Subacts and Graded Morita Reduction

A second, more directly applicable criterion comes from full homogeneous corners. Let $B \subseteq A$. We say that B is C-full if

$$\mathrm{Sat}_C(B) = A.$$

When A is finite, define

$$p_B = \sum_{b \in B} b \in L_K(\Gamma_C(A)).$$

This is a homogeneous idempotent of degree zero.

Theorem 7.1 (Full-subact criterion). Let A be a finite S-act and $C \subseteq S$ finite. If $B \subseteq A$ is C-full, then

$$p_B L_K(\Gamma_C(A)) p_B$$

is graded Morita equivalent to $L_K(\Gamma_C(A))$. If the corner $p_B L_K(\Gamma_C(A)) p_B$ is isomorphic as a graded algebra to $L_K(\Gamma_D(B))$ for some induced semigroup-act graph $\Gamma_D(B)$, then $L_K(\Gamma_D(B))$ is graded Morita equivalent to $L_K(\Gamma_C(A))$.

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Proof. Let $E = \Gamma_C(A)$. The ideal generated by p_B is the graded ideal generated by B. By Proposition 4.3, this ideal corresponds to $\mathrm{Sat}_C(B)$. Since B is C-full, $\mathrm{Sat}_C(B) = A = E^0$, and hence the ideal generated by p_B is all of $L_K(E)$. Thus p_B is a full homogeneous idempotent. The standard graded Morita theorem for full homogeneous corners gives the graded Morita equivalence between $L_K(E)$ and $p_B L_K(E) p_B$. The final assertion follows by composing this equivalence with the assumed graded isomorphism $p_B L_K(E) p_B \cong L_K(\Gamma_D(B))$.

The criterion is especially useful when B is a hereditary full subset: then paths beginning in B remain in B, and the corner is often the Leavitt path algebra of the restricted subgraph.

8. Finite Monogenic Acts

Let $S = \langle t \rangle$ be a monogenic monoid and let A be a finite left S-act. The action is determined by a single function

$$\tau: A \to A, \qquad \tau(a) = ta.$$

The graph $\Gamma_{\{t\}}(A)$ is a functional directed graph: every vertex emits exactly one edge.

Lemma 8.1 (Functional decomposition). Every finite monogenic act decomposes into weakly connected components, each consisting of one directed cycle with finite rooted trees feeding into the cycle.

Proof. Starting at any $a \in A$, the finite sequence $a, \tau(a), \tau^2(a), \ldots$ must eventually repeat. The first repetition creates a directed cycle. Every element that reaches this cycle lies on a finite directed tree directed toward the cycle. Two distinct directed cycles cannot lie in the same weak component of a functional graph, because following the unique outgoing edge from any vertex gives a unique eventual cycle. Hence each weak component has exactly one cycle with rooted trees feeding into it.

Define the cycle core by

$$\mathrm{Core}(A) = \{a \in A : a \text{ lies on a directed cycle of } \tau\}.$$

Lemma 8.2 (Core is hereditary and full). For every finite monogenic act A, the subset $\mathrm{Core}(A)$ is $\{t\}$-hereditary and $\{t\}$-full.

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Proof. If a lies on a cycle, then $\tau(a) = ta$ also lies on the same cycle, so $\mathrm{Core}(A)$ is $\{t\}$-hereditary. By Lemma 8.1, every vertex eventually maps into the unique cycle of its component. The saturation operation for $C = \{t\}$ adds a vertex whenever its t-image has already been added. Starting from the core, repeated saturation therefore adds every vertex in every tree feeding into the core. Hence $\mathrm{Sat}_{\{t\}}(\mathrm{Core}(A)) = A$.

Theorem 8.3 (Monogenic reduction). Let A be a finite monogenic act. Then $L_K\left(\Gamma_{\{t\}}(A)\right)$ is graded Morita equivalent to $L_K\left(\Gamma_{\{t\}}(\mathrm{Core}(A))\right)$. If the cycle lengths of the components of $\mathrm{Core}(A)$ are $p_1, \dots, p_r$, then

$$K_0^{\mathrm{gr}}\left(L_K\left(\Gamma_{\{t\}}(A)\right)\right) \cong \oplus_{i=1}^{r} \mathbb{Z}[x, x^{-1}]/(1 - x^{p_i})$$

as a $\mathbb{Z}[x, x^{-1}]$-module.

Proof. By Lemma 8.2, $\mathrm{Core}(A)$ is full. Apply Theorem 7.1 with $B = \mathrm{Core}(A)$. Since the core is hereditary, the corner $p_B L_K\left(\Gamma_{\{t\}}(A)\right) p_B$ is the Leavitt path algebra of the restricted core graph. Thus $L_K\left(\Gamma_{\{t\}}(A)\right)$ is graded Morita equivalent to $L_K\left(\Gamma_{\{t\}}(\mathrm{Core}(A))\right)$. The core graph is a disjoint union of directed cycles $C_{p_1}, \dots, C_{p_r}$. For a directed cycle of length p, the action matrix is the p-cycle permutation matrix, and Theorem 5.2 gives the module $\mathbb{Z}[x, x^{-1}]/(1 - x^p)$. Taking the finite disjoint union gives the stated direct sum.

Corollary 8.4 (A computable sufficient classification). Two finite monogenic acts whose cycle cores have the same multiset of cycle lengths have graded Morita equivalent Leavitt path algebras. If their ordered K_0^{gr}-modules are not isomorphic, then their Leavitt path algebras are not graded Morita equivalent.

Proof. The first assertion follows from Theorem 8.3 because both algebras reduce to the same disjoint union of cycle algebras up to ordering of the components. The second assertion follows from invariance of ordered K_0^{gr} under graded Morita equivalence. We do not claim here that the multiset of cycle lengths is a complete invariant in every possible graded-categorical refinement; the statement is restricted to the reduction obtained above.

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9. Examples and Counterexamples

9.1 A DIRECTED 3-CYCLE ACT

Let $A=\{a_1,a_2,a_3\}$ and let the monogenic action be
$$ta_1=a_2,\qquad ta_2=a_3,\qquad ta_3=a_1.$$
Then $\Gamma_{\{t\}}(A)$ is the directed 3-cycle and
$$M=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}.$$
The talented monoid has generators $a_i(k)$ and relations
$$a_1(k)=a_2(k+1),\qquad a_2(k)=a_3(k+1),\qquad a_3(k)=a_1(k+1).$$
Thus
$$K_0^{\text{gr}}\cong\mathbb{Z}[x,x^{-1}]/(1-x^3).$$
Indeed, the three relations imply $e_1=xe_2=x^2e_3=x^3e_1$, hence $(1-x^3)e_1=0$; conversely $e_2=x^{-1}e_1$ and $e_3=x^{-2}e_1$.

9.2 TAIL ATTACHED TO A LOOP

Let $A=\{a_1,a_2,a_3\}$ with
$$ta_1=a_2,\qquad ta_2=a_3,\qquad ta_3=a_3.$$
The vertex a_3 is the cycle core, a loop of length 1. The subset $\{a_3\}$ is $\{t\}$-hereditary and full because saturation adds a_2 and a_1. By Theorem 7.1,
$$L_K\left(\Gamma_{\{t\}}(A)\right)\sim_{\text{grMorita}}L_K(\text{loop})\cong K[x,x^{-1}].$$
The K_0^{gr} presentation also collapses to $\mathbb{Z}[x,x^{-1}]/(1-x)$, as predicted by Theorem 8.3.

9.3 NON-ISOMORPHIC ACTS WITH GRADED MORITA EQUIVALENT ALGEBRAS

Let $A=\{a_0,a_1\}$ with $ta_0=a_1$ and $ta_1=a_1$. Let $B=\{b_0,b_1,b_2\}$ with $tb_0=b_1$, $tb_1=b_2$ and $tb_2=b_2$. These acts are not isomorphic because they have different cardinalities and tree depths. Nevertheless, each has a one-vertex loop as its cycle core. It follows from Theorem 8.3 that
$$L_K\left(\Gamma_{\{t\}}(A)\right)\sim_{\text{grMorita}}K[x,x^{-1}]\sim_{\text{grMorita}}L_K\left(\Gamma_{\{t\}}(B)\right).$$
Thus graded Morita equivalence is coarser than isomorphism of acts.

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9.4 $\boldsymbol{K_0^{\mathrm{gr}}}$ SEPARATES CYCLE LENGTHS

Let A be the 2-cycle monogenic act and B the 3-cycle monogenic act. By Theorem 8.3,

$$K_0^{\mathrm{gr}}\left(L_K\left(\Gamma_{\{t\}}(A)\right)\right) \cong \mathbb{Z}[x,x^{-1}]/(1-x^2),$$
$$K_0^{\mathrm{gr}}\left(L_K\left(\Gamma_{\{t\}}(B)\right)\right) \cong \mathbb{Z}[x,x^{-1}]/(1-x^3).$$

These modules are not isomorphic as $\mathbb{Z}[x,x^{-1}]$-modules because their annihilator ideals are respectively $(1-x^2)$ and $(1-x^3)$, which are distinct. Hence the corresponding Leavitt path algebras are not graded Morita equivalent.

9.5 A NON-LIFTABILITY COUNTEREXAMPLE

Let E be the graph with one vertex v and two loops. It can be realized as $\Gamma_C(A)$ with $A=\{v\}$ and $C=\{c_1,c_2\}$, where both elements act by $c_i v = v$. Thus, every vertex emits $|C|=2$ edges. Let F be obtained by adjoining a new source w and a single edge $f: w \to v$, while keeping the two loops at v. The subset $\{v\}$ is hereditary and full in F, since saturation adds w. Therefore $L_K(E)$ is graded Morita equivalent to $L_K(F)$ by the full-corner argument. However, F cannot be of the form $\Gamma_D(B)$ for any finite nonempty D under the fixed C-induced convention of this paper, because a graph $\Gamma_D(B)$ has the same outdegree $|D|$ at every vertex, counted with multiplicity. In F, the vertex v emits two edges and w emits one edge. Hence this graph-theoretic graded Morita equivalence does not lift to a graded Morita equivalence inside the class of fixed-generator semigroup-act graphs. This example shows that the act-derived framework is genuinely restrictive.

10. Conclusion and Open Problems

10.1 CONCLUSION

This paper develops act-theoretic criteria for graded Morita equivalence of Leavitt path algebras associated with semigroup-act graphs. The dictionary between hereditary and saturated graph subsets and C-hereditary and C-saturated act subsets provides an intrinsic description of graded ideals. For finite acts, the talented monoid and K_0^{gr} can be computed directly from the action matrix. The principal criterion is act-induced aligned shift equivalence, while the full-subact criterion yields practical reductions through full homogeneous corners. For finite monogenic acts, the Leavitt path algebra is graded Morita equivalent to the algebra of the cycle core.

These results also delineate the limits of the approach. The class of fixed-C semigroup-act graphs is not closed under graph-theoretic graded Morita equivalence. This limitation is not a defect; rather, it reflects the additional algebraic structure encoded by the underlying semigroup action.

10.2 OPEN PROBLEMS

1. Find necessary and sufficient act-theoretic conditions for graded Morita equivalence of $L_K\big(\Gamma_C(A)\big)$ and $L_K\big(\Gamma_D(B)\big)$.

2. Determine precisely which graph moves preserving graded Morita equivalence lift to natural operations on semigroup acts.

3. Classify finite acts over finite regular semigroups using action matrices, K_0^{gr} and talented monoids. Work on regular right uniform semigroups suggests that semigroup-internal regularity conditions may lead to tractable subclasses (Hosseinzadeh Alikhalaji, Sedaghatjoo and Roueentan, 2021).

4. Determine whether ordered K_0^{gr} together with the talented monoid is complete for selected classes of semigroup-act graphs beyond finite monogenic acts.

5. Develop explicit algorithms for deciding act-induced aligned shift equivalence when A, B, C and D are finite.

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