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دراسة حول المتباينات الديناميكية من النمط النسبي على المقاييس الزمنية

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المستلخص:

يتضمن هذا العمل نظائر النطاق الزمني لبعض المتباينات التكاملية الكلاسيكية ذات الأشكال النسبية مع الدوال الموزونة، وذلك لتوحيد النتائج في حسابات التفاضل والتكامل المتصلة والمتقطعة في متباينة واحدة ذات نطاق عام يُسمى النطاق الزمني $\mathbb{T}$، والذي يُعرَّف بأنه مجموعة فرعية مغلقة غير خالية من الأعداد الحقيقية $\mathbb{R}$. يكمن دور إثبات أي متباينة على النطاقات الزمنية في تجنب إثباتها مرتين (مرة في الحالة المتصلة ومرة أخرى في الحالة المتقطعة). تُعطي نتائجنا، في الحالة المتصلة، امتدادًا للنتائج السابقة المُثبتة في المراجع [7] و[10] و[12]، بينما يُمكننا أيضًا الحصول على نظائر متقطعة أخرى مختلفة. بالإضافة إلى ذلك، يُمكننا الحصول على متباينات أخرى في حساب النطاق الزمني(الكوانتم).

الكلمات المفتاحية:متباينات من نوع النسبة؛ قاعدة السلسلة على المقاييس الزمنية؛ متباينة جنسن.

A STUDY ON RATIO-TYPE DYNAMIC INEQUALITIES ON TIME SCALES

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Abstract:

This work includes the time scale analogues of some classical integral inequalities of ratio forms with weighted functions to unify the results in the continuous and discrete calculi in a single inequality with general domain called a time scale $\mathbb{T}$ which is defined by a nonempty closed subset of the real numbers $\mathbb{R}$. The role of proving any inequality on time scales, is to avoid proving this inequality twice (one in the continuous case and the other in the discrete case). Our results (when $\mathbb{T}=\mathbb{R}$), give the extension of the earlier results established in [7], [10], and [12], while $\mathbb{T}=\mathbb{N}$, give other different discrete analogues. Additionally, we can obtain other different analogues in the quantum time scale calculus.

Keywords: Ratio-type inequalities, Chain rule on time scales, Jensen's inequality.

1. Introduction

Ratio-type integral inequalities have an important class of inequalities in mathematical analysis due to their ability to estimate relationships between integral expressions through quotient structures. These inequalities provide useful tools for getting qualitative properties of solutions of differential, difference, and dynamic equations. Moreover, they play a significant role in establishing bounds, stability criteria, and comparison results.

The ratio type integral inequality was established in [10]. Let $\varepsilon, \delta \in \mathbb{R}, \delta > \varepsilon$, and $\xi, \omega, \chi \in C([\varepsilon, \delta], (0, \infty))$, such that for $\tau \in [\varepsilon, \delta]$,

$$\left(\omega(\tau)-\omega(\vartheta)\right)\left(\frac{\xi(\vartheta)}{\chi(\vartheta)}-\frac{\xi(\tau)}{\chi(\tau)}\right)\geq 0. \qquad (1)$$

Then

$$\left(\int_{\varepsilon}^{\delta}\xi(\tau)\mathrm{d}\tau\right)\left(\int_{\varepsilon}^{\delta}\chi(\vartheta)\omega(\vartheta)\mathrm{d}\vartheta\right)$$
$$\geq\left(\int_{\varepsilon}^{\delta}\chi(\tau)\mathrm{d}\tau\right)\left(\int_{\varepsilon}^{\delta}\xi(\vartheta)\omega(\vartheta)\mathrm{d}\vartheta\right). \qquad (2)$$

In addition, they showed that if $\varepsilon,\delta\in\mathbb{R},\delta>\varepsilon,p\geq 1$, and $\xi,\chi\in C([\varepsilon,\delta],(0,\infty))$ with $\xi(x)\leq\chi(x)$ for all $x\in[\varepsilon,\delta]$, where the function ξ is increasing and $\frac{\xi}{\chi}$ is decreasing, then,

$$\frac{\int_{\varepsilon}^{\delta}\xi(\tau)\mathrm{d}\tau}{\int_{\varepsilon}^{\delta}\chi(\tau)\mathrm{d}\tau}\geq\frac{\int_{\varepsilon}^{\delta}\xi^{p}(\vartheta)\mathrm{d}\vartheta}{\int_{\varepsilon}^{\delta}\chi^{p}(\vartheta)\mathrm{d}\vartheta}, \qquad (3)$$

and if Ξ is a convex function on $(0,\infty)$ with $\Xi(0)=0$, then,

$$\frac{\int_{\varepsilon}^{\delta}\Xi(\xi(\vartheta))\mathrm{d}\vartheta}{\int_{\varepsilon}^{\delta}\Xi(\chi(\vartheta))\mathrm{d}\vartheta}\leq\frac{\int_{\varepsilon}^{\delta}\xi(\vartheta)\mathrm{d}\vartheta}{\int_{\varepsilon}^{\delta}\chi(\vartheta)\mathrm{d}\vartheta}. \qquad (4)$$

A distinct ratio-type integral inequality was proved in [12] and showed that if $\alpha,\ell\in\mathbb{R},\ell>\alpha$ and $\phi,\Xi\in C([\alpha,\ell],(0,\infty))$, where $\phi\Xi,\phi/\Xi$ are both nondecreasing, then

$$\frac{\int_{\alpha}^{b}\frac{1}{\phi(\vartheta)}\mathrm{d}\vartheta}{\int_{\alpha}^{b}\frac{1}{\Xi(\vartheta)}\mathrm{d}\vartheta}\geq\frac{\int_{\alpha}^{b}\Xi(\vartheta)\mathrm{d}\vartheta}{\int_{\alpha}^{b}\phi(\vartheta)\mathrm{d}\vartheta}. \qquad (5)$$

The generalized form of inequality (5) was proved in [7] by adding the weight function λ and showed that if $\alpha,\ell\in\mathbb{R},\ell>\alpha$, $\phi,\psi\in C([\alpha,\ell],(0,\infty))$, where $\phi\psi,\phi/\psi$ are both nondecreasing and $\lambda \in C([\alpha, \ell], (0, \infty))$ is a weight function, then

$$\frac{\int_\alpha^b \frac{\lambda(\vartheta)}{\phi(\vartheta)} \mathrm{d}\vartheta}{\int_\alpha^b \frac{\lambda(\vartheta)}{\psi(\vartheta)} \mathrm{d}\vartheta} \geq \frac{\int_\alpha^b \psi(\vartheta)\lambda(\vartheta)\mathrm{d}\vartheta}{\int_\alpha^b \phi(\vartheta)\lambda(\vartheta)\mathrm{d}\vartheta}. \tag{6}$$

and also under the same conditions of ϕ and ψ, he established that

$$\frac{(b-\alpha)\int_\alpha^b \psi^2(\vartheta)\mathrm{d}\vartheta}{\int_\alpha^b \frac{\psi(\vartheta)}{\phi(\vartheta)} \mathrm{d}\vartheta} \leq \int_\alpha^b \phi(\vartheta)\psi(\vartheta)\mathrm{d}\vartheta \leq \frac{(b-\alpha)\int_\alpha^b \phi^2(\vartheta)\mathrm{d}\vartheta}{\int_\alpha^b \frac{\phi(\vartheta)}{\psi(\vartheta)} \mathrm{d}\vartheta}. \tag{7}$$

Motivation: In this work, we generalize the classical integral inequalities in the continuous case which were established in [7], [10], and [12], to include a general domain a time scale $\mathbb{T}$, which is defined as a nonempty closed subset of the real numbers $\mathbb{R}$. The new dynamic inequalities on time scales are considered unified versions of the integral inequalities and their corresponding discrete analogue. Therefore, as a special case of our results (when $\mathbb{T}=\mathbb{R}$), we get these classical integral inequalities, while $\mathbb{T}=\mathbb{N}$, we get their new corresponding discrete analogues.

The time scale theory unifies the continuous calculous and discrete calculus as in [9]. Therefore, the classical integral and discrete inequalities can be unified and generalized in a single dynamic inequality with a general domain “a time scale $\mathbb{T}$”. Additionally, by using the definition of a time scale $\mathbb{T}$, one can get other inequalities in several classes, like; $\mathbb{R}, \mathbb{N}$, $h\mathbb{N}$ for $h > 0$, $[1,4]$, $[1,3] \cup [2,4]$, and $q^{\mathbb{N}}$ for $q > 1$. For more

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information about the dynamic inequality on time scales, see the papers [1, 2, 3, 4, 11, 13, 14, 16, 18, 19, 20] and the monograph [1]. For explanation, the authors in [8] showed that if $p > 1, \alpha < p-1$ and ξ is measurable and nonnegative for $\tau \in (0,\infty)$, then

$$\int_0^\infty \tau^\alpha \left(\frac{1}{\tau}\int_0^\tau \xi(\vartheta)\mathrm{d}\vartheta\right)^p \mathrm{d}\tau \leq \left(\frac{p}{p-\alpha-1}\right)^p \int_0^\infty \tau^\alpha \xi^p(\tau)\mathrm{d}\tau. \qquad (8)$$

The time scale version of inequality (8) was introduced in [17] to show that if $p, \gamma > 1, f \in C_{\mathrm{rd}}([\varrho,\infty)_{\mathbb{T}}, \mathbb{R}^+)$, $\varrho \geq 0$ and $\mathrm{E}(\tau) = \int_\varrho^\tau f(s)\Delta s$ for any $\tau \in [\varrho,\infty)_{\mathbb{T}}$, then

$$\int_\varrho^\infty \frac{(\mathrm{E}^\sigma(\tau))^p}{(\sigma(\tau)-\varrho)^\gamma}\Delta\tau \leq \left(\frac{p}{\gamma-1}\right)^p \int_\varrho^\infty \frac{(\sigma(\tau)-\varrho)^{\gamma(p-1)}}{(\tau-\varrho)^{p(\gamma-1)}} f^p(\tau)\Delta\tau. \qquad (9)$$

If the time scale $\mathbb{T}$ is chosen as $\mathbb{R}$, relation (9) becomes the well-known integral inequality in (8). Selecting $\mathbb{N}$ produces the discrete version, while taking $q^{\mathbb{N}_0}$ with $q > 1$ leads to its counterpart in quantum calculus.

This work develops time-scale extensions of several ratio-type integral inequalities, specifically those labeled (2), (3), (4), (6), and (7). The proofs rely on properties of convex functions, differentiation rules adapted to time scales, and certain algebraic results from dynamic equations.

The structure of the paper is as follows. Section 2 gathers preliminary results and auxiliary statements on time scales that

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العدد الثامن والأربعون
٢٠٢٦م /١٤٤٨هـ

are needed later. Section 3 contains the principal theorems, along with further explanations, observations, corollaries, and illustrative examples.

2. AUXILIARY LEMMAS

Instead of repeating the basic facts of time scales and time scale notation, we refer the reader to the monograph [5], which provides a comprehensive overview of the theory. Here, we simply state a few basic theorems that will be essential in Section 3.

A time scale $\mathbb{T}$ is defined by a nonempty closed subset of the real numbers $\mathbb{R}$. For $t \in \mathbb{T}$, we define the forward jump operator $\sigma: \mathbb{T} \to \mathbb{T}$ and the backward jump operator $\rho: \mathbb{T} \to \mathbb{T}$ as follows:

$$\sigma(\tau) = \inf\{\vartheta \in \mathbb{T}: \vartheta > \tau\}$$

and

$$\rho(\tau) = \sup\{\vartheta \in \mathbb{T}: \vartheta < \tau\},$$

while the graininess function $\mu: \mathbb{T} \to [0, \infty)$ is given by $\mu(\tau) = \sigma(\tau) - \tau$. The set $\mathbb{T}^\kappa$ is given by

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T} \setminus (\rho(\sup\mathbb{T}), \sup\mathbb{T}] & \text{if } \sup\mathbb{T} < \infty; \\ \mathbb{T} & \text{if } \sup\mathbb{T} = \infty. \end{cases}$$

For $a, b \in \mathbb{T}$, the interval $[a, b]_{\mathbb{T}}$ on time scales is

defined by $[a, b] \cap \mathbb{T}$.

Example 1: Let $\mathbb{T}$ be a time scale. Then, we have the following.

1. If $\mathbb{T} = \mathbb{R}$, then $\sigma(\tau) = \tau$, $\rho(\tau) = \tau$, and $\mu(\tau) = 0$ for any $\tau \in \mathbb{R}$.
2. If $\mathbb{T} = \mathbb{N}$, then $\sigma(\tau) = \tau + 1$, $\rho(\tau) = \tau - 1$, and

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العدد الثامن والأربعون
٢٠٢٦م/١٤٤٨هـ

$\mu(\tau) = 1$ for any $\tau \in \mathbb{N}$.

3. If $\mathbb{T} = \mathrm{q}^{\mathbb{N}}$ for $q > 1$, then $\sigma(\tau) = \mathrm{q}\tau$, $\rho(\tau) = \tau/\mathrm{q}$, and $\mu(\tau) = (\mathrm{q}-1)\tau$ for any $\tau \in \mathrm{q}^{\mathbb{N}}$, $q > 1$.

Definition 2.1. (Delta derivative, see [5]).

Assume that $f: \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^{\mathrm{k}}$.

We define $f^{\Delta}(\mathrm{t})$ to be the number, provided it exists.

For any $\varepsilon > 0$, there is a neighborhood U of t,

$U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$, such that

$$\left|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]\right| \leq \varepsilon\, |\sigma(t) - s| \forall s \in U, \quad s \neq \sigma(t).$$

In this case, we say $f^{\Delta}(\mathrm{t})$ is the delta or Hilger derivative of f at t. We say that f is delta or Hilger differentiable, shortly differentiable, in $\mathbb{T}^{\mathrm{k}}$ if $f^{\Delta}(\mathrm{t})$ exists for all $t \in \mathbb{T}^{\mathrm{k}}$. The function $f^{\Delta}: \mathbb{T}^{\mathrm{k}} \to \mathbb{R}$ is said to be the delta derivative or Hilger derivative, shortly derivative, of f in $\mathbb{T}^{\mathrm{k}}$.

Theorem 2.1 (See [5]). Let $\omega, \upsilon: \mathbb{T} \to \mathbb{R}$ be differentiable at $\tau \in \mathbb{T}^{k}$. Then we have

(1) The product $\omega\upsilon: \mathbb{T} \to \mathbb{R}$ is differentiable at τ and the "product rule"

$$\begin{aligned}(\omega\upsilon)^{\Delta}(\tau) &= \omega^{\Delta}(\tau)\upsilon(\tau) + \omega\big(\sigma(\tau)\big)\upsilon^{\Delta}(\tau) \\ &= \omega(\tau)\upsilon^{\Delta}(\tau) + \omega^{\Delta}(\tau)\upsilon\big(\sigma(\tau)\big), \qquad (10)\end{aligned}$$

holds.

(2) If $\upsilon(\tau)\upsilon(\sigma(\tau)) \neq 0$, then we have the quotient $\omega/\upsilon: \mathbb{T} \to \mathbb{R}$ is differentiable at τ and the "quotient rule"

JOBS مجلة العلوم الأساسية
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العدد الثامن والأربعون
٢٠٢٦م /١٤٤٨هـ

$$\left(\frac{\omega}{\upsilon}\right)^{\Delta}(\tau)=\frac{\omega^{\Delta}(\tau)\upsilon(\tau)-\omega(\tau)\upsilon^{\Delta}(\tau)}{\upsilon(\tau)\upsilon\big(\sigma(\tau)\big)}, \qquad (11)$$

holds.

Theorem 2.2 (See [5]). If $\omega^{\Delta}\geq 0$ ($\omega^{\Delta}\leq 0$), then ω is nondecreasing (nonincreasing).

Theorem 2.3 (Chain Rule, see [6, Theorem 1.87]). Assume $v:\mathbb{R}\to\mathbb{R}$ is continuous, $v:\mathbb{T}\to\mathbb{R}$ is delta differentiable on $\mathbb{T}^{\kappa}$, and $u:\mathbb{R}\to\mathbb{R}$ is continuously differentiable. Then there exists c in the real interval $[t,\sigma(t)]$ with

$$(u\circ v)^{\Delta}(t)=u'\big(v(c)\big)v^{\Delta}(t). \qquad (12)$$

Definition 2.2. (See [5])

A function $f:\mathbb{T}\to\mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f:\mathbb{T}\to\mathbb{R}$ is denoted by C_{rd} or $C_{\mathrm{rd}}(\mathbb{T})$ or $C_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$.

Theorem 2.4 (See [5, Theorem 1.77]). If $a,b,c\in\mathbb{T},\alpha,\beta\in\mathbb{R}$ and $u,v\in C_{\mathrm{rd}}([a,b]_{\mathbb{T}},\mathbb{R})$, then

(1) $\int_a^b[\alpha u(t)+\beta v(t)]\Delta t$

$$=\alpha\int_a^b u(t)\Delta t+\beta\int_a^b v(t)\Delta t,$$

(2) $\int_a^b u(t)\Delta t=-\int_b^a u(t)\Delta t,$

(3) the integration by parts formula

JOBS مجلة العلوم الأساسية
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العدد الثامن والأربعون
٢٠٢٦م /١٤٤٨هـ

$$\int_a^b u(\zeta)v^{\Delta}(\zeta)\Delta\zeta$$
$$= [u(\zeta)v(\zeta)]_a^b - \int_a^b u^{\Delta}(\zeta)v^{\sigma}(\zeta)\Delta\zeta, \qquad (13)$$

holds.

Theorem 2.5 (See [5]). If $\xi \in C_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$ and $\tau \in \mathbb{T}$, then

$$\int_{\tau}^{\sigma(\tau)} \xi(\vartheta)\Delta\vartheta = \mu(\tau)\xi(\tau), \qquad (14)$$

where the graininess function $\mu: \mathbb{T} \to [0,\infty)$, is given by $\mu(\tau) = \sigma(\tau) - \tau,\ \ for\ \tau \in \mathbb{T}$.

Lemma 2.1 (Hölder's inequality, see [5]). If $\varepsilon,\delta \in \mathbb{T}$ and $\lambda,\varpi \in \mathrm{C_{rd}}([\varepsilon,\delta]_{\mathbb{T}},\mathbb{R}^{+})$, then

$$\int_{\varepsilon}^{\delta} \lambda(\zeta)\varpi(\zeta)\Delta\zeta \le \left[\int_{\varepsilon}^{\delta} \lambda^{\gamma}(\zeta)\Delta\zeta\right]^{\frac{1}{\gamma}} \left[\int_{\varepsilon}^{\delta} \varpi^{\nu}(\zeta)\Delta\zeta\right]^{\frac{1}{\nu}}, \qquad (15)$$

where $\gamma > 1$ and $1/\gamma + 1/\nu = 1$.

Lemma 2.2 (Jensen's inequality, see [5]). If $\varepsilon,\delta \in \mathbb{T}, c,d \in \mathbb{R}, \lambda \in C_{\mathrm{rd}}([\varepsilon,\delta]_{\mathbb{T}},(c,d))$, and $\Xi \in C((c,d),\mathbb{R})$ is convex, then

$$\Xi\left(\frac{\int_{\varepsilon}^{\delta} \lambda(\zeta)\Delta\zeta}{\delta-\varepsilon}\right) \le \frac{\int_{\varepsilon}^{\delta} \Xi\big(\lambda(\zeta)\big)\Delta\zeta}{\delta-\varepsilon}. \qquad (16)$$

The function $\Xi \in C((c,d),\mathbb{R})$ is convex provided that

$$\Xi(\beta x + (1-\beta)\tau) \le \beta\Xi(x) + (1-\beta)\Xi(\tau), \qquad (17)$$

4

$\forall\beta \in [0,1], \forall x,\tau \in (c,d)$.

JOBS مجلة العلوم الأساسية
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Online-ISSN 2791-3279
العدد الثامن والأربعون
٢٠٢٦م /١٤٤٨هـ

3. MAIN RESULTS

Throughout this section, we assume all integrals considered exist and finite. We now establish the time scale analogue of inequality (6).

Theorem 3.1. Assume $\alpha, b \in \mathbb{T}, b > \alpha, \phi, \psi \in C_{rd}([\alpha, b]_{\mathbb{T}}, (0, \infty))$, where $\phi\psi, \phi/\psi$ are both nondecreasing and $\lambda \in C_{rd}([\alpha, b]_{\mathbb{T}}, (0, \infty))$ is a weight function. Then

$$\frac{\int_\alpha^b \frac{\lambda(\tau)}{\phi(\tau)}\Delta\tau}{\int_\alpha^b \frac{\lambda(\tau)}{\psi(\tau)}\Delta\tau} \geq \frac{\int_\alpha^b \psi(\tau)\lambda(\tau)\Delta\tau}{\int_\alpha^b \phi(\tau)\lambda(\tau)\Delta\tau}. \tag{18}$$

Proof. Define a function Ξ, where for $x \in [\alpha, b]$,

$$\Xi(x): = \left(\int_\alpha^x \phi(\zeta)\lambda(\zeta)\Delta\zeta\right)\left(\int_\alpha^x \frac{\lambda(\zeta)}{\phi(\zeta)}\Delta\zeta\right) - \left(\int_\alpha^x \psi(\zeta)\lambda(\zeta)\Delta\zeta\right)\left(\int_\alpha^x \frac{\lambda(\zeta)}{\psi(\zeta)}\Delta\zeta\right). \tag{19}$$

By applying the product rule of differentiation on time scales, we get

$$\Xi^\Delta(x) = \left[\frac{\lambda(x)}{\phi(x)}\int_\alpha^x \phi(\tau)\lambda(\tau)\Delta\tau + \lambda(x)\phi(x)\int_\alpha^{\sigma(x)} \frac{\lambda(\tau)}{\phi(\tau)}\Delta\tau - \frac{\lambda(x)}{\psi(x)}\int_\alpha^x \psi(\tau)\lambda(\tau)\Delta\tau - \lambda(x)\psi(x)\int_\alpha^{\sigma(x)} \frac{\lambda(\tau)}{\psi(\tau)}\Delta\tau\right]$$

JOBS مجلة العلوم الأساسية
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العدد الثامن والأربعون
٢٠٢٦م /١٤٤٨هـ

$$= \lambda(x)\left[\frac{1}{\phi(x)}\int_{\alpha}^{x}\phi(\tau)\lambda(\tau)\Delta\tau + \phi(x)\int_{\alpha}^{\sigma(x)}\frac{\lambda(\tau)}{\phi(\tau)}\Delta\tau - \frac{1}{\psi(x)}\int_{\alpha}^{x}\psi(\tau)\lambda(\tau)\Delta\tau - \psi(x)\int_{\alpha}^{\sigma(x)}\frac{\lambda(\tau)}{\psi(\tau)}\Delta\tau\right]. \quad (20)$$

Employing Theorem 2.5, we find that

$$\int_{\alpha}^{\sigma(x)}\frac{\lambda(\zeta)}{\phi(\zeta)}\Delta\zeta = \int_{\alpha}^{x}\frac{\lambda(\zeta)}{\phi(\zeta)}\Delta\zeta + \int_{x}^{\sigma(x)}\frac{\lambda(\zeta)}{\phi(\zeta)}\Delta\zeta = \int_{\alpha}^{x}\frac{\lambda(\zeta)}{\phi(\zeta)}\Delta\zeta + \mu(x)\frac{\lambda(x)}{\phi(x)} \quad (21)$$

and

$$\int_{\alpha}^{\sigma(x)}\frac{\lambda(\zeta)}{\psi(\zeta)}\Delta\zeta = \int_{\alpha}^{x}\frac{\lambda(\zeta)}{\psi(\zeta)}\Delta\zeta + \int_{x}^{\sigma(x)}\frac{\lambda(\zeta)}{\psi(\zeta)}\Delta\zeta = \int_{\alpha}^{x}\frac{\lambda(\zeta)}{\psi(\zeta)}\Delta\zeta + \mu(x)\frac{\lambda(x)}{\psi(x)}, \quad (22)$$

hence

$$\phi(x)\int_{\alpha}^{\sigma(x)}\frac{\lambda(\tau)}{\phi(\tau)}\Delta\tau - \psi(x)\int_{\alpha}^{\sigma(x)}\frac{\lambda(\zeta)}{\psi(\zeta)}\Delta\zeta = \phi(x)\int_{\alpha}^{x}\frac{\lambda(\zeta)}{\phi(\zeta)}\Delta\zeta - \psi(x)\int_{\alpha}^{x}\frac{\lambda(\zeta)}{\psi(\zeta)}\Delta\zeta. \quad (23)$$

Substituting (23) into (20), it follows that

$$\Xi^{\Delta}(x) = \lambda(x)\int_{\alpha}^{x}\left[\frac{\phi(\zeta)}{\phi(x)} + \frac{\phi(x)}{\phi(\zeta)} - \frac{\psi(\zeta)}{\psi(x)} - \frac{\psi(x)}{\psi(\zeta)}\right]\lambda(\zeta)\Delta\zeta$$

JOBS
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العدد الثامن والأربعون
٢٠٢٦م /١٤٤٨هـ

$$= \frac{\lambda(x)}{\phi(x)} \int_{\alpha}^{x} \left[\left(\frac{\phi(x)}{\psi(x)} - \frac{\phi(\zeta)}{\psi(\zeta)} \right) (\phi(x)\psi(x) \right.$$
$$\left. - \phi(\zeta)\psi(\zeta)) \right] \frac{\lambda(\zeta)}{\phi(\zeta)} \Delta\zeta. \qquad (24)$$

Taking in account the two nondecreasing functions $\frac{\phi}{\psi}$ and $\phi\psi$, we have for $\zeta \leq x$, that

$$\frac{\phi(x)}{\psi(x)} \geq \frac{\phi(\zeta)}{\psi(\zeta)} \text{ and } \phi(x)\psi(x) \geq \phi(\zeta)\psi(\zeta),$$

i.e.,

$$\frac{\phi(x)}{\psi(x)} - \frac{\phi(\zeta)}{\psi(\zeta)} \geq 0 \text{ and } \phi(x)\psi(x) - \phi(\zeta)\psi(\zeta) \geq 0, \qquad (25)$$

thus (24) gives

$$\Xi^{\Delta}(x) = \frac{\lambda(x)}{\phi(x)} \int_{\alpha}^{x} \left[\left(\frac{\phi(x)}{\psi(x)} - \frac{\phi(\zeta)}{\psi(\zeta)} \right) (\phi(x)\psi(x) - \phi(\zeta)\psi(\zeta)) \right]$$
$$\times \frac{\lambda(\zeta)}{\phi(\zeta)} \Delta\zeta \geq 0, \qquad (26)$$

and then by applying Theorem 2.2, then the function Ξ is nondecreasing. Therefore, by using the definition of the function Ξ, we observe that for $x \in [a, b]$,

$$\left(\int_{\alpha}^{x} \phi(\zeta)\lambda(\zeta)\Delta\zeta \right) \left(\int_{\alpha}^{x} \frac{\lambda(\zeta)}{\phi(\zeta)} \Delta\zeta \right)$$
$$\geq \left(\int_{\alpha}^{x} \psi(\zeta)\lambda(\zeta)\Delta\zeta \right) \left(\int_{\alpha}^{x} \frac{\lambda(\zeta)}{\psi(\zeta)} \Delta\zeta \right). \qquad (27)$$

Setting $x = b$ in the inequality (27), it follows that

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$$\left(\int_{\alpha}^{b} \phi(\tau)\lambda(\tau)\Delta\tau\right)\left(\int_{\alpha}^{b} \frac{\lambda(\tau)}{\phi(\tau)}\Delta\tau\right)$$
$$\geq \left(\int_{\alpha}^{b} \psi(\tau)\lambda(\tau)\Delta\tau\right)\left(\int_{\alpha}^{b} \frac{\lambda(\tau)}{\psi(\tau)}\Delta\tau\right), \qquad (28)$$

i.e.,

$$\frac{\int_{\alpha}^{b} \frac{\lambda(\tau)}{\phi(\tau)}\Delta\tau}{\int_{\alpha}^{b} \frac{\lambda(\tau)}{\psi(\tau)}\Delta\tau} \geq \frac{\int_{\alpha}^{b} \psi(\tau)\lambda(\tau)\Delta\tau}{\int_{\alpha}^{b} \phi(\tau)\lambda(\tau)\Delta\tau}, \qquad (29)$$

and this informs the validity of (18).

Remark 3.1. It is obvious that Theorem 3.1 is also satisfied when $\phi\psi$ and ϕ/ψ are both nonincreasing functions which is clear from (26).

Remark 3.2. Taking $\mathbb{T} = \mathbb{R}$, then the dynamic inequality (18) reduces to the integral inequality (6) proved in [7].

Corollary 3.1. Assume $\mathbb{T} = \mathbb{N}, \alpha, b \in \mathbb{N}, b > \alpha, \phi, \psi, \lambda$ are positive sequences, where $\phi\psi, \phi/\psi$ are both nondecreasing and λ is a weight sequence. Then,

$$\frac{\sum_{\tau=\alpha}^{b-1} \frac{\lambda(\tau)}{\phi(\tau)}}{\sum_{\tau=\alpha}^{b-1} \frac{\lambda(\tau)}{\psi(\tau)}} \geq \frac{\sum_{\tau=\alpha}^{b-1} \psi(\tau)\lambda(\tau)}{\sum_{\tau=\alpha}^{b-1} \phi(\tau)\lambda(\tau)}. \qquad (30)$$

Corollary 3.2. Determining $\mathbb{T} = q^{\mathbb{N}}$ with $q > 1, \alpha = q^m, b = q^n, m, n \in \mathbb{N}, n > m, \phi, \psi, \lambda$ are positive sequences, where $\phi\psi, \phi/\psi$ are both nondecreasing and λ is a weight sequence. Then,

JOBS مجلة العلوم الأساسية
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$$\frac{\sum_{\tau=m}^{n-1} q^\tau \frac{\lambda(q^\tau)}{\phi(q^\tau)}}{\sum_{\tau=m}^{n-1} q^\tau \frac{\lambda(q^\tau)}{\psi(q^\tau)}} \geq \frac{\sum_{\tau=m}^{n-1} q^\tau \psi(q^\tau)\lambda(q^\tau)}{\sum_{\tau=m}^{n-1} q^\tau \phi(q^\tau)\lambda(q^\tau)}. \qquad (31)$$

Example 2. Determining $\mathbb{T} = \mathbb{R}, \alpha = 0, b = 2, \lambda(\zeta) = \zeta^4, \phi(\zeta) = \zeta^2$, and $\psi(\zeta) = \zeta$, then the left side of (18) for the continuous case, becomes

$$\frac{\int_\alpha^b \frac{\lambda(\tau)}{\phi(\tau)}\Delta\tau}{\int_\alpha^b \frac{\lambda(\tau)}{\psi(\tau)}\Delta\tau} = \frac{\int_0^2 \tau^2\, d\tau}{\int_0^2 \tau^3\, d\tau} = 0.6666667 \qquad (32)$$

and the right side of (18) gives

$$\frac{\int_\alpha^b \psi(\tau)\lambda(\tau)\Delta\tau}{\int_\alpha^b \phi(\tau)\lambda(\tau)\Delta\tau} = \frac{\int_0^2 \tau^5\, d\tau}{\int_0^2 \tau^6\, d\tau} = 0.583333 \qquad (33)$$

and this shows the validity of (18) for the continuous case.

Example 3. Choosing $\mathbb{T} = \mathbb{N}, \alpha = 1, b = 4, \lambda(\vartheta) = \vartheta^3, \phi(\vartheta) = \vartheta^2$, and $\psi(\vartheta) = \vartheta$, then the left side of (30), gives

$$\frac{\sum_{\tau=\alpha}^{b-1} \frac{\lambda(\tau)}{\phi(\tau)}}{\sum_{\tau=\alpha}^{b-1} \frac{\lambda(\tau)}{\psi(\tau)}} = \frac{\sum_{\tau=1}^{3} \tau}{\sum_{\tau=1}^{3} \tau^2} = 0.428571 \qquad (34)$$

and the right side of (30) becomes

$$\frac{\sum_{\tau=\alpha}^{b-1} \psi(\tau)\lambda(\tau)}{\sum_{\tau=\alpha}^{b-1} \phi(\tau)\lambda(\tau)} = \frac{\sum_{\tau=1}^{3} \tau^4}{\sum_{\tau=1}^{3} \tau^5} = 0.355072 \qquad (35)$$

and this shows the validity of (30).

Example 4. Taking $\mathbb{T} = 2^{\mathbb{N}}, q = 2, m = 1, n = 5, \lambda(\vartheta) = \vartheta + 2, \phi(\vartheta) = \vartheta^3$, and $\psi(\vartheta) = \vartheta$, then the left side of (31) becomes

JOBS مجلة العلوم الأساسية
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Online-ISSN 2791-3279
العدد الثامن والأربعون
٢٠٢٦م /١٤٤٨هـ

$$\frac{\sum_{\tau=m}^{n-1} q^\tau \frac{\lambda(q^\tau)}{\phi(q^\tau)}}{\sum_{\tau=m}^{n-1} q^\tau \frac{\lambda(q^\tau)}{\psi(q^\tau)}} = \frac{\sum_{\tau=1}^{4} \frac{2^\tau+2}{2^{2\tau}}}{\sum_{\tau=1}^{4} (2^\tau+2)} = 0.04215 \qquad (36)$$

and the right side of (31) becomes

$$\frac{\sum_{\tau=m}^{n-1} q^\tau\psi(q^\tau)\lambda(q^\tau)}{\sum_{\tau=m}^{n-1} q^\tau\phi(q^\tau)\lambda(q^\tau)} = \frac{\sum_{\tau=1}^{4} 2^{2\tau}(2^\tau+2)}{\sum_{\tau=1}^{4} 2^{4\tau}(2^\tau+2)}$$
$$= 0.004385 \qquad (37)$$

and this confirms that (31) holds.

Theorem 3.2. Assume $\alpha, b \in \mathbb{T}, b > \alpha$, and $\phi, \psi \in C_{rd}([\alpha, b]_{\mathbb{T}}, (0, \infty))$ such that $\phi\psi, \phi/\psi$ are both nondecreasing. Then

$$\frac{(b-\alpha)\int_\alpha^b \psi^2(\tau)\Delta\tau}{\int_\alpha^b \frac{\psi(\tau)}{\phi(\tau)}\Delta\tau} \le \int_\alpha^b \phi(\tau)\psi(\tau)\Delta\tau$$
$$\le \frac{(b-\alpha)\int_\alpha^b \phi^2(\tau)\Delta\tau}{\int_\alpha^b \frac{\phi(\tau)}{\psi(\tau)}\Delta\tau}. \qquad (38)$$

Proof. Applying Theorem 3.2 with $\lambda = \phi$, we get

$$\frac{b-\alpha}{\int_\alpha^b \frac{\phi(\zeta)}{\psi(\zeta)}\Delta\zeta} \ge \frac{\int_\alpha^b \psi(\zeta)\phi(\zeta)\Delta\zeta}{\int_\alpha^b \phi^2(\zeta)\Delta\zeta}, \qquad (39)$$

and by setting $\lambda = \psi$, we obtain

$$\frac{\int_\alpha^b \frac{\psi(\zeta)}{\phi(\zeta)}\Delta\zeta}{b-\alpha} \ge \frac{\int_\alpha^b \psi^2(\zeta)\Delta\zeta}{\int_\alpha^b \phi(\zeta)\psi(\zeta)\Delta\zeta}. \qquad (40)$$

Combining (39) and (40), we have

$$\frac{(b-\alpha)\int_\alpha^b \psi^2(\zeta)\Delta\zeta}{\int_\alpha^b \frac{\psi(\zeta)}{\phi(\zeta)}\Delta\zeta} \leq \int_\alpha^b \phi(\zeta)\psi(\zeta)\Delta\zeta$$

$$\leq \frac{(b-\alpha)\int_\alpha^b \phi^2(\zeta)\Delta\zeta}{\int_\alpha^b \frac{\phi(\zeta)}{\psi(\zeta)}\Delta\zeta}, \qquad (41)$$

and this satisfies (38).

Remark 3.3. Choosing $\mathbb{T}=\mathbb{R}$, then the dynamic inequality (38) reduces to the integral inequality (7) proved in [7].

Theorem 3.3. Let $a, b \in \mathbb{T}, b > a$, and $f, g, h \in C_{rd}([a,b]_{\mathbb{T}}, (0,\infty))$, with

$$(g(\tau)-g(\zeta))\left(\frac{f(\zeta)}{h(\zeta)}-\frac{f(\tau)}{h(\tau)}\right) \geq 0. \qquad (42)$$

Then

$$\left(\int_a^b f(\tau)\Delta\tau\right)\left(\int_a^b h(\zeta)g(\zeta)\Delta\zeta\right)$$
$$\geq \left(\int_a^b h(\tau)\Delta\tau\right)\left(\int_a^b f(\zeta)g(\zeta)\Delta\zeta\right). \qquad (43)$$

Proof. Denote

$$A := \left(\int_a^b f(\tau)\Delta\tau\right)\left(\int_a^b h(\zeta)g(\zeta)\Delta\zeta\right)$$
$$-\left(\int_a^b h(\tau)\Delta\tau\right)\left(\int_a^b f(\zeta)g(\zeta)\Delta\zeta\right)\#$$
$$= \int_a^b \int_a^b g(\zeta)[f(\tau)h(\zeta)-h(\tau)f(\zeta)]\Delta\tau\Delta\zeta. \qquad (44)$$

JOBS مجلة العلوم الأساسية
Journal of Basic Science
Print -ISSN 2306-5249
Online-ISSN 2791-3279
العدد الثامن والأربعون
٢٠٢٦م /١٤٤٨هـ

Since the last integrals are definite integrals, we can write

$$A := \left(\int_a^b f(\zeta)\Delta\zeta\right)\left(\int_a^b h(\tau)g(\tau)\Delta\tau\right)$$
$$-\left(\int_a^b h(\zeta)\Delta\zeta\right)\left(\int_a^b f(\tau)g(\tau)\Delta\tau\right)\#$$
$$= \int_a^b \int_a^b g(\tau)[f(\zeta)h(\tau) - h(\zeta)f(\tau)]\Delta\tau\Delta\zeta. \qquad (45)$$

Adding (44) and (45), one can get

$$A = \frac{1}{2}\int_a^b \int_a^b \big(g(\tau) - g(\zeta)\big)$$
$$\times [f(\zeta)h(\tau) - h(\zeta)f(\tau)]\Delta\tau\Delta\zeta,$$
$$= \frac{1}{2}\int_a^b \int_a^b h(\tau)h(\zeta)\big(g(\tau) - g(\zeta)\big)\left[\frac{f(\zeta)}{h(\zeta)}\right.$$
$$\left. - \frac{f(\tau)}{h(\tau)}\right]\Delta\tau\Delta\zeta, \qquad (46)$$

thus from (42), we find that $A \geq 0$ and then (44) gives

$$\left(\int_a^b f(\tau)\Delta\tau\right)\left(\int_a^b h(\zeta)g(\zeta)\Delta\zeta\right)$$
$$\geq \left(\int_a^b h(\tau)\Delta\tau\right)\left(\int_a^b f(\zeta)g(\zeta)\Delta\zeta\right),$$

and this satisfies (43).

Remark 3.4. Determining $\mathbb{T} = \mathbb{R}$, then the inequality (43) reduces to (2) established in [10].

JOBS مجلة العلوم الأساسية
Journal of Basic Science
Print -ISSN 2306-5249
Online-ISSN 2791-3279
العدد الثامن والأربعون
٢٠٢٦م /١٤٤٨هـ

Corollary 3.3. Assume $\mathbb{T}=\mathbb{N}, a, b \in \mathbb{N}, b>a$, and f, g, h are positive sequences, with

$$\left(g(\tau)-g(\zeta)\right)\left(\frac{f(\zeta)}{h(\zeta)}-\frac{f(\tau)}{h(\tau)}\right) \geq 0. \tag{47}$$

Then,

$$\left(\sum_{\tau=a}^{b-1} f(\tau)\right)\left(\sum_{\zeta=a}^{b-1} h(\zeta)g(\zeta)\right) \geq \left(\sum_{\tau=a}^{b-1} h(\tau)\right)\left(\sum_{\zeta=a}^{b-1} f(\zeta)g(\zeta)\right). \tag{48}$$

Example 5. Taking $\mathbb{T}=\mathbb{N}, a=1, b=5, g(\tau)=\tau^2, f(\tau)=\tau$, and $h(\tau)=\tau^3$. Then, the left side of (48) gives

$$\left(\sum_{\tau=a}^{b-1} f(\tau)\right)\left(\sum_{\zeta=a}^{b-1} h(\zeta)g(\zeta)\right)=\left(\sum_{\tau=1}^{4} \tau\right)\left(\sum_{\zeta=1}^{4} \zeta^5\right) = 13000 \tag{49}$$

and the right side of (48) yields

$$\left(\sum_{\tau=a}^{b-1} h(\tau)\right)\left(\sum_{\zeta=a}^{b-1} f(\zeta)g(\zeta)\right)=\left(\sum_{\tau=1}^{4} \tau^3\right)\left(\sum_{\zeta=1}^{4} \zeta^3\right) = 10000 \tag{50}$$

and this shows the correctness of (48).

Theorem 3.4. Assume $a, b \in \mathbb{T}, b>a, p \geq 1$, and $f, h \in$

$C_{rd}([a,b]_{\mathbb{T}},(0,\infty))$ with $f(\tau)\leq h(\tau)$ for all $\tau\in[a,b]_{\mathbb{T}}$, such that the function f is increasing and $\frac{f}{h}$ is decreasing. Then,

$$\frac{\int_a^b f(\tau)\Delta\tau}{\int_a^b h(\tau)\Delta\tau}\geq\frac{\int_a^b f^p(\zeta)\Delta\zeta}{\int_a^b h^p(\zeta)\Delta\zeta}. \tag{51}$$

Proof. Since f is increasing and $p\geq 1$, then f^{p-1} is also increasing. In this case, we have the increasing function f^{p-1} and the decreasing function $\frac{f}{h}$, which satisfy the condition (42) when $g=f^{p-1}$, thus by applying Theorem 3.3, especially, the inequality (43), we get

$$\frac{\int_a^b f(\tau)\Delta\tau}{\int_a^b h(\tau)\Delta\tau}\geq\frac{\int_a^b f^p\zeta\Delta\zeta}{\int_a^b h(\zeta)f^{p-1}(\zeta)\Delta\zeta}. \tag{52}$$

Taking into the account that $p\geq 1$ and $f(\zeta)\leq h(\zeta)$ for all $\zeta\in[a,b]_{\mathbb{T}}$, the inequality (52) gives

$$\frac{\int_a^b f(\tau)\Delta\tau}{\int_a^b h(\tau)\Delta\tau}\geq\frac{\int_a^b f^p(\zeta)\Delta\zeta}{\int_a^b h^p(\zeta)\Delta\zeta},$$

which is (51).

Remark 3.5. Determining $\mathbb{T}=\mathbb{R}$, then the inequality (51) reduces to (3) proved in [10].

In the following, we generalize Theorem 3.4 to include the convex function ϕ by replacing $f^p(\tau), p\geq 1$ with $\phi(f(\tau))$.

Theorem 3.5. Assume $a,b\in\mathbb{T}, b>a$, and $f,h\in C_{rd}([a,b]_{\mathbb{T}},(0,\infty))$ with $f(\zeta)\leq h(\zeta)$ $\forall\zeta\in[a,b]_{\mathbb{T}}$, where the function f is increasing and $\frac{f}{h}$ is decreasing. If ϕ is a convex function on $(0,\infty)$ with $\phi(0)=0$, then

$$\frac{\int_a^b \phi(f(\zeta))\Delta\zeta}{\int_a^b \phi(h(\zeta))\Delta\zeta} \le \frac{\int_a^b f(\zeta)\Delta\zeta}{\int_a^b h(\zeta)\Delta\zeta}. \tag{53}$$

Proof. By employing the definition of the convex function ϕ in (17), where $\phi(0)=0$, then by setting $\tau=0$ and $x=\zeta$, we have for $\zeta>0$,

$$\phi(\beta\zeta) \le \beta\phi(\zeta), \forall\beta \in [0,1], \tag{54}$$

i.e.,

$$\frac{\phi(\beta\zeta)}{\beta\zeta} \le \frac{\phi(\zeta)}{\zeta}, \forall\beta \in [0,1]. \tag{55}$$

Taking $\vartheta=\beta\zeta$ implies $\vartheta\le\zeta$, so, in this case, we have

$$\frac{\phi(\vartheta)}{\vartheta} \le \frac{\phi(\zeta)}{\zeta} \quad \text{for } \vartheta \le \zeta, \tag{56}$$

which indicates that $\frac{\phi(\zeta)}{\zeta}$ is increasing and then for $f(\zeta) \le h(\zeta)$, we observe that

$$\frac{\phi(f(\zeta))}{f(\zeta)} \le \frac{\phi(h(\zeta))}{h(\zeta)}. \tag{57}$$

Since f and $\frac{\phi(\zeta)}{\zeta}$ are increasing, their composition also $\frac{\phi(f(\zeta))}{f(\zeta)}$ becomes increasing. Thus for $\zeta \le b$,

$$\frac{\phi(f(\zeta))}{f(\zeta)} \le \frac{\phi(f(b))}{f(b)}. \tag{58}$$

Combining the two inequalities (57) and (58), it follows that

JOBS
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Journal of Basic Science
Print -ISSN 2306-5249
Online-ISSN 2791-3279
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$$\frac{\int_a^b \phi(f(\zeta))\Delta\zeta}{\int_a^b \phi(h(\zeta))\Delta\zeta} = \frac{\int_a^b f(\zeta)\frac{\phi(f(\zeta))}{f(\zeta)}\Delta\zeta}{\int_a^b h(\zeta)\frac{\phi(h(\zeta))}{h(\zeta)}\Delta\zeta}$$

$$\leq \frac{\int_a^b f(\zeta)\frac{\phi(f(\zeta))}{f(\zeta)}\Delta\zeta}{\int_a^b h(\zeta)\frac{\phi(f(\zeta))}{f(\zeta)}\Delta\zeta}. \qquad (59)$$

Since $\frac{f}{h}(.)$ is decreasing and $\frac{\phi(f(.))}{f(.)}$ is increasing, then the condition (42) in Theorem 3.3, by setting $g(\zeta) = \frac{\phi(f(\zeta))}{f(\zeta)}$, holds and then in this case, we obtain from (43) that

$$\frac{\int_a^b f(\zeta)\frac{\phi(f(\zeta))}{f(\zeta)}\Delta\zeta}{\int_a^b h(\zeta)\frac{\phi(f(\zeta))}{f(\zeta)}\Delta\zeta} \leq \frac{\int_a^b f(\zeta)\Delta\zeta}{\int_a^b h(\zeta)\Delta\zeta}. \qquad (60)$$

Substituting (60) into (59), gives

$$\frac{\int_a^b \phi(f(\zeta))\Delta\zeta}{\int_a^b \phi(h(\zeta))\Delta\zeta} \leq \frac{\int_a^b f(\zeta)\Delta\zeta}{\int_a^b h(\zeta)\Delta\zeta},$$

which is (53).

Remark 3.6. Determining $\mathbb{T} = \mathbb{R}$, the dynamic inequality (53) reduces to the classical integral (4) proved in [10].

Corollary 3.4. Assume $\mathbb{T} = \mathbb{N}, a, b \in \mathbb{N}, b > a$, and f, h are positive sequences with $f(\zeta) \leq h(\zeta)$ $\forall \zeta \in [a, b]$, where

JOBS مجلة العلوم الأساسية
Journal of Basic Science
Print -ISSN 2306-5249
Online-ISSN 2791-3279
العدد الثامن والأربعون
٢٠٢٦م /١٤٤٨هـ

$\{f(n)\}_a^b$ is increasing and $\left\{\frac{f}{h}(n)\right\}_a^b$ is decreasing. If ϕ is a convex function on $(0,\infty)$ with $\phi(0)=0$, then

$$\frac{\sum_{\vartheta=a}^{b-1}\phi(f(\vartheta))}{\sum_{\vartheta=a}^{b-1}\phi(h(\vartheta))}\leq\frac{\sum_{\vartheta=a}^{b-1}f(\vartheta)}{\sum_{\vartheta=a}^{b-1}h(\vartheta)}. \tag{61}$$

Example 6. Taking $\mathbb{T}=\mathbb{N}, a=1, b=4, \phi(\vartheta)=\vartheta^2, f(\vartheta)=\vartheta$, and $h(\vartheta)=\vartheta^3$. Then, the left side of (61) gives

$$\frac{\sum_{\vartheta=a}^{b-1}\phi(f(\vartheta))}{\sum_{\vartheta=a}^{b-1}\phi(h(\vartheta))}=\frac{\sum_{\vartheta=1}^{3}\vartheta^2}{\sum_{\vartheta=1}^{3}\vartheta^6}=0.01763 \tag{62}$$

and the right side of (61) becomes

$$\frac{\sum_{\zeta=a}^{b-1}f(\zeta)}{\sum_{\zeta=a}^{b-1}h(\zeta)}=\frac{\sum_{\zeta=1}^{3}\zeta}{\sum_{\zeta=1}^{3}\zeta^3}=0.1667. \tag{63}$$

Thus, the inequality (61) is correct.

Conclusion

In this paper, we generalized the classical ratio-type integral inequalities established by Chesneau, Makanin, and Liu et al. in the unified framework (time scales) to include the other analogues of these results in the discrete and quantum cases, which are considered new and interesting to the reader. These dynamic inequalities were proved for the monotonic and convex functions. Furthermore, as special cases of our results, when $\mathbb{T}=\mathbb{R}$, we get the classical integral inequalities of these authors mentioned above, while $\mathbb{T}=\mathbb{N}$, we obtain new corresponding discrete analogues, and also when $\mathbb{T}=q^{\mathbb{N}}$ for q>1, we establish the quantum analogues. Additionally, we can get other new analogues of these inequalities in different classes as special cases of a time scale $\mathbb{T}$, like; $h\mathbb{N}$, $h > 0$, [1,2], and [3,4]∪ [5,6]. Furthermore, in the future work, we are interested in establishing the dynamic ratio-type inequalities on diamond alpha time scales to include delta and nabla time scales in the same inequality. As we seek to establish new ratio-type inequalities involving C-monotonic functions on time scales.

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Print -ISSN 2306-5249
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العدد الثامن والأربعون
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