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توسيع لطريقة سينك–غاليركين مع تقنية التعامد لحل مسائل القيم الحدودية من الرتبة السادسة

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المستخلص:

لا يزال الحل العددي الدقيق لمسائل القيم الحدودية (BVPs) من الرتبة السادسة يمثل تحديًا كبيرًا؛ نظرًا لاشتمالها على مشتقات من رتب عليا واحتمالية وجود نقاط شاذة (singularities) في نطاق الحل. تظهر هذه المسائل بشكل متكرر في مجالات ديناميكا الموائع، ونظرية المرونة، والفيزياء الفلكية. في هذا العمل، نقدم إطارًا عدديًا قويًا وعالي الدقة يعتمد على طريقة سينك–غاليركين (Sinc–Galerkin) لحل مسائل القيم الحدودية الخطية وغير الخطية من الرتبة السادسة. تتميز الطريقة المقترحة بإدراجها لتقنية تعامد (orthogonalization) منهجية، والتي تحوّل النظام الخطي الناتج، الذي يكون عادةً كثيفًا وغير متماثل، إلى صيغة متماثلة وموجبة التحديد (symmetric positive–definite). يضمن هذا التحويل استقرارية الحسابات ويسهّل عملية إيجاد الحل بكفاءة عبر استخدام تفكيك تشوليسكي (Cholesky factorisation). علاوة على ذلك، يتم حساب العناصر الأساسية للنظام المتقطع باستخدام تربيع غاوس–ليجاندر (Gauss–Legendre quadrature) ذي الرتبة العالية، مما يساهم في تحقيق الدقة الطيفية (spectral accuracy) الشاملة للطريقة.

لتقييم أداء الطريقة، قمنا بتطبيقها على مسائل قياسية ذات حلول تحليلية معروفة. أظهرت الدراسات المقارنة مع أحدث الطرق المعتمدة على كثيرات الحدود، مثل طريقة بي–سبلاين بتروف–غاليركين (B–spline Petrov–Galerkin) وطريقة التجميع لليجاندر (Legendre collocation)، أن طريقتنا تحقق دقة فائقة تفوق الطرق الأخرى بعدة مراتب عشرية (orders of magnitude)، خاصة في التقاط الحلول الملساء والمتناقصة أسيًا.

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تؤكد النتائج العددية أيضًا معدل التقارب الأسي (exponential convergence rate) النظري لطريقة سينك–غاليركين، والذي يأخذ الصيغة $O\left(\exp(-c\sqrt{N})\right)$، حيث يمثل N عدد نقاط التجميع (collocation points) لدالة سينك. إن سلوك التقارب هذا، مقترنًا بعمومية الإطار المقترح، يرسخ هذه الطريقة كأداة فعّالة لحل فئة واسعة من مسائل القيم الحدودية ذات الرتب العليا وبدقة وموثوقية عاليتين.

الكلمات المفتاحية : مسائل القيم الحدودية من الرتبة السادسة، طريقة سينك–غاليركين، التقارب الأسي، الدقة الطيفية، التعامد، مسائل القيم الحدودية غير الخطية.

An Extension of the Sinc-Galerkin Method with OrthogonalisationTechnique for Solving Sixth-Order Boundary Value Problems

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Abstract

Because of their high order of derivatives and possibility for singularities in the solution domain, the exact numerical solution of sixth-order boundary value problems (BVPs), which often arise in fluid dynamics, elasticity theory, and astrophysics, remains a difficult issue. We present in this work a strong and very accurate numerical framework based on the Sinc-Galerkin approach for solving both linear and nonlinear sixth-order BVPs. The suggested method distinguishes itself by including a methodical orthogonalisation technique that converts the resultant dense and non-symmetric system into a symmetric positive-definite form. This guarantees computational stability and, by use of Cholesky factorisation, facilitates effective solutions. High-order Gauss-Legendre quadrature evaluates the fundamental elements of the discrete system, therefore adding to the general spectrum accuracy of the approach. We apply the technique to benchmark problems with known analytic solutions in order to assess performance. Comparative studies against state-of-the-art polynomial-based schemes, including B-spline Petrov-Galerkin and Legendre collocation methods, show that our method achieves superior accuracy by several orders of magnitude,

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٢٠٢٥م /١٤٤٧هـ

especially in capturing smooth and exponentially declining solutions. The numerical results further confirm the theoretical
exponential convergence rate of the Sinc-Galerkin scheme, of the form $O\left(\exp(-c\sqrt{N})\right)$, where N denotes the number of Sinc collocation points. This convergence behavior, coupled with the generality of the proposed framework, establishes the method as a powerful tool for solving a broad class of high-order BVPs with high fidelity.

Keywords: Sixth-Order Boundary Value Problems, Sinc-Galerkin Method, Exponential Convergence, Spectral Accuracy, Orthogonalization, Nonlinear BVPs.

1. Introduction

Models spanning a broad spectrum of physical and technical events depend on boundary value problems (BVPs). Especially in elasticity theory, fluid dynamics, and beam theory, higher-order BVPs, including sixth-order issues, surface. Especially when singularities arise close to the domain borders (Agarwal, 1986), these analytically complex and computationally stiff problems sometimes make conventional finite difference or finite element approaches useless or wasteful in precisely capturing solution behaviour.

Sinc-based methods represent a potential family of numerical techniques that have shown amazing effectiveness in addressing these obstacles. These approaches apply the cardinal sinc function:

$$sinc(x) = \frac{\sin(\pi x)}{\pi x}, x \neq 0; sinc(0) = 1$$

which, with appropriate transformations, has an exponential convergence rate and thus great approximating qualities on infinite or semi-infinite intervals. Third: Sinc functions are globally specified, unlike traditional basis functions, which are localised; thus, they are particularly useful for managing solitary or unbounded domains .

A variation including the Galerkin projection into the Sinc basis framework is the Sinc-Galerkin approach. A great range of issues has seen successful use of this method. For handling end singularities, Sinc-collocation was applied to solve eighth-order BVPs with double exponential (DE) mapping (Qiu et al., 2022, p. 114136); non-classical Sinc formulations have

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demonstrated great accuracy in physiological models (Mohammadi et al., 2022, pp. 1941–1967) .
Al-Haliji made a major contribution by addressing the application of the Sinc-Galerkin technique to fourth-order pseudo-Poisson equations, therefore exposing an underlying weakness in the formulation of the conventional linear systems. The author organised the linear system using an orthogonalising method to fix both numerical and theoretical inefficiencies. This improved method attained exponential convergence of the type:

$$\| u(x) - u_N(x) \| = O\left(\exp(-c\sqrt{N})\right), \text{as } N \to \infty$$

where $u(x)$ is the exact solution, $u_N(x)$ is the numerical approximation, N is the number of basis functions, and $c > 0$ is a constant depending on the problem's analytic structure.
Building on this basis, the present study expands this approach to sixth-order boundary value issues, which are intrinsically more difficult and less investigated in the literature. The main results of this work are:

• Derivation of a consistent Galerkin formulation for generic sixth-order linear BVPs.

• Modification of the orthogonalising approach to sustain exponential convergence.

• Presentation of numerical tests verifying the correctness and durability of the technique.

This expansion not only shows the adaptability of the Sinc-Galerkin method but also offers a methodical road for applied mathematics' high-order singular issue solution.

2. Research Objectives

The primary objectives of this study are:

• To derive the discrete Sinc-Galerkin system for sixth-order BVPs, incorporating the orthogonalisation technique to ensure a stable and efficient solution.

- To implement the proposed method and numerically validate its performance, analysing its accuracy and confirming the exponential rate of convergence on selected test problems

3. Methodology

The proposed numerical scheme is developed within a rigorous theoretical framework. This section outlines the class of problems under investigation and reviews the essential mathematical theory of the Sinc function, which forms the basis of our method.

3.1. THEORETICAL FRAMEWORK

3.1.1. THE GENERAL SIXTH-ORDER BOUNDARY VALUE PROBLEM

This research addresses the numerical solution of the general linear sixth-order boundary value problem (Chandrashekhar, 1961). Such equations are of significant interest as they model various physical phenomena. A notable application is found in astrophysics; it was determined that when an infinite horizontal layer of fluid is heated from below and subjected to the action of rotation, instability sets in through ordinary convection, which is governed by a sixth-order ordinary differential equation.

The general form of the differential equation considered in this paper is (El-Gamel, 2004, pp. 785–802; Olaiya et al., 2020, pp. 1-15):

$$p_0(t)\,v^{(6)}(t) + \sum_{i=1}^{6} p_i\,(t)\,v^{(6-i)}(t) = q(t), a < t < b$$

This equation is subject to a set of specified boundary conditions. The functions are assumed continuous on the interval, and the existence and uniqueness of solutions for such problems have been thoroughly discussed in the literature (Agarwal, 1986; Martinez et al., 2021, pp. 1-12). While obtaining analytical solutions is generally not possible, numerous researchers have developed numerical methods to approximate them. These methods include, but are not limited to, Adomian decomposition (Wazwaz, 2001, pp. 311–325), variational approaches, various spline techniques (Siddiqi & Akram, 2006, pp. 798–811; Siddiqi & Akram, 2008, pp. 288–301;

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٢٠٢٥م /١٤٤٧هـ

Viswanadham & Kiranmayi Ch, 2017, pp. 6301–6308), and the homotopy perturbation method.

3.1.2. PRELIMINARIES OF THE SINC FUNCTION AND CARDINAL SERIES

The Sinc numerical method offers a powerful alternative for such problems, especially those with singular behaviour at the boundaries (Lund & Bowers, 1992; Parand et al., 2020, p. 2221; Stenger, 1993). It is constructed upon the properties of the cardinal function, which provides an exact reconstruction formula for functions within a specific class.

The cornerstone of the method is the **Sinc function**, defined as:

$$sinc(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

For a function $F(x)$ defined on the real line R, its approximation is given by the **cardinal series**, a sum of translated Sinc functions weighted by the function's values at discrete nodes:

$$C(F,h)(x) = \sum_{k=-\infty}^{\infty} F(kh)\, S(k,h)(x)$$

where $h > 0$ is the step size, and $S(k,h)(x)$ are the translated basis functions defined as (Lund & Bowers, 1992; Parand et al., 2020, p. 2221; Stenger, 1993):

$$S(k,h)(x) = sinc\left(\frac{x-kh}{h}\right)$$

A key advantage of the Sinc function basis is its **orthogonality** with respect to the discrete inner product, satisfying:

$$S(k,h)(jh) = \delta_{kj}$$

where δ_{kj} is the **Kronecker delta**. The most significant feature of Sinc-based methods is their **exponential convergence rate** (Parand et al., 2020, p. 2221; Stenger, 1993). For functions that are analytic in a strip region of the complex plane, the approximation error decays as:

$$\| F(x) - C(F,h)(x) \| = O\left(\exp\left(-\frac{c}{\sqrt{h}}\right)\right)$$

where $c > 0$ is a constant depending on the analyticity region (Siddiqi & Akram, 2006, pp. 798–811). This rapid convergence is particularly advantageous for problems with endpoint singularities, offering a significant improvement over the polynomial convergence rates of classical methods.

3.2 Derivation of the Discrete Galerkin System

The transformation of the continuous differential equation into a discrete algebraic system is the central task of the numerical method. This is accomplished using the Petrov-Galerkin method, a powerful and flexible framework within the broader class of finite element methods (Golub & Van Loan, 2013). The core principle is to enforce a weighted orthogonality condition upon the residual of the differential equation .

The first step is to approximate the unknown solution $u(x)$ using a finite cardinal series expansion over $N = M_1 + M_2 + 1$ nodes :

$$u(x) \approx u_N(x) = \sum_{k=-M_1}^{M_2} u_k \ S(k,h)(x)$$

where u_k are the unknown coefficients to be determined. The Galerkin principle then dictates that the residual, $L(u_N) - f$, must be made orthogonal to each basis function $S(j,h)(x)$ (Siddiqi & Akram, 2006, pp. 798–811). This condition is formally expressed through the inner product:

$$\langle L(u_N) - f, S(j,h) \rangle_w = 0, j = -M_1, \dots, M_2$$

The use of a weighted inner product, $\langle \cdot,\cdot \rangle_w$, is a standard technique in Sinc methods to ensure the convergence of the resulting integrals (Parand et al., 2020, p. 2222), especially after a conformal map has been applied to transform the problem from a finite interval (a, b) to the real line R (Parand et al., 2020, p. 2222).

A direct evaluation of the terms, such as $\left\langle u_N^{(6)}, S(j,h) \right\rangle_w$, would involve integrating high-order derivatives of Sinc functions, a process that is both analytically and computationally intensive. The elegant approach in Sinc theory is to employ integration by parts repeatedly (Doha et al., 2010, pp. 800–819). This strategically transfers the derivatives from the basis function

$S(k,h)(x)$ within $u_N(x)$ to the test function $S(j,h)(x)$, resulting in integrals that are much simpler to approximate.

This procedure gives rise to a set of well-defined $N \times N$ matrices, denoted $I^{(m)}$, whose entries effectively represent the action of the derivative operator $\frac{d^m}{dx^m}$ in the discrete Sinc space (Parand et al., 2020, p. 2222). For instance, the well-known symmetric matrices for even derivatives have a defined structure (Siddiqi & Akram, 2006, pp. 798–811). The primary contribution of this work is the explicit formulation of the discrete system for the sixth-order problem. Following the operational calculus of Sinc methods, the required derivative matrices can be constructed from the lower-order ones. In particular, the sixth-order derivative matrix is formed by the composition of the second-order operator:

$$I^{(6)} \approx \left(I^{(2)}\right)^3$$

Similarly, the odd-order derivative matrices, such as $I^{(1)}, I^{(3)}$, and $I^{(5)}$, are anti-symmetric and can also be pre-computed (Qiu et al., 2022, p. 114136).

By assembling these matrix components, the entire differential equation is discretized into the matrix system (Siddiqi & Akram, 2006, pp. 798–811):

$$Au = G$$

where $u = \left[u_{-M_1}, \ldots, u_{M_2}\right]^T$ is the vector of unknown coefficients. The $N \times N$ system matrix A is a linear combination of the derivative matrices and the coefficient functions:

$$A = \sum_{m=0}^{6} D\,(p_{6-m})\, I^{(m)}$$

Here, $D(p_i)$ are diagonal matrices with the values of the function $p_i(x)$ at the Sinc nodes. The vector G on the right-hand side is derived from the inner product involving the forcing function $f(x)$. The resulting matrix A is typically dense and non-symmetric. To address this, the Orthogonalisationtechnique, previously developed for fourth-order problems, will be adapted and applied to this more complex system to ensure a robust and efficient solution (Oyedepo et al., 2024).

3.3. System Assembly and the OrthogonalisationFramework

The discretization process detailed in the preceding section culminates in the formation of a linear algebraic system, $Au = G$. We now turn our attention to the structural properties of the system matrix A and the robust numerical strategy required for its solution.

3.3.1. Analysis of the System Matrix

The assembled system matrix is given by:

$$A = \sum_{m=0}^{6} D\,(p_{6-m})\,I^{(m)}$$

This matrix possesses characteristics that present significant computational challenges. Specifically:

- **Non-Symmetry:** The presence of discrete operators for odd-order derivatives (i.e., $I^{(1)}, I^{(3)}, I^{(5)}$), which are anti-symmetric, coupled with the coefficient matrices $D(p_i)$, results in a system matrix A that is, in general, non-symmetric. This precludes the use of highly efficient iterative and direct solvers designed for symmetric matrices, such as the **Conjugate Gradient** method or **Cholesky factorisation** (Golub & Van Loan, 2013).
- **Condition Number:** Matrices arising from the discretization of differential operators, especially for large numbers of nodes N, can be **ill-conditioned**. In such cases, small perturbations in the right-hand side vector G may lead to large deviations in the computed solution vector u, rendering the direct numerical inversion $u = A^{-1}G$ unreliable.

3.3.2. The OrthogonalisationTechnique as a Regularization Method

To address these structural deficiencies, we adapt a specialized **Orthogonalisationtechnique**, which can be viewed as a form of **system regularization**. The objective is not merely to solve the system, but to transform it into an equivalent, well-posed problem with superior numerical properties Oyedepo et al., 2024).

The core idea is to construct a new system of the form:

$$Cu = H$$

where the matrix C is **symmetric** and **positive-definite (SPD)**. Such matrices guarantee the existence of a unique solution and allow the use of the most stable and efficient algorithms in numerical linear algebra.

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العدد الثلاثون
٢٠٢٥م/ ١٤٤٧هـ

The transformation begins by identifying the principal symmetric part of the operator A, denoted A_{sym}, which is composed of the even-ordered derivative matrices known to be symmetric:

$$A_{\text{sym}} = I^{(6)} + D(p_2)I^{(4)} + D(p_4)I^{(2)} + D(p_6)I^{(0)}$$

We then define the new regularized system as:

$$Cu = H$$

with:

$$C = A_{\text{sym}}^T A_{\text{sym}} + \gamma R \; H = A_{\text{sym}}^T G$$

Analyzing the properties of C:

- The term $A_{\text{sym}}^T A_{\text{sym}}$ is, by construction, **symmetric** and **positive semi-definite**.

- The matrix R is a chosen **symmetric positive-definite regularization matrix**, typically taken as the identity matrix I or another smoothness-enforcing operator like $I^{(2)}$.

- The scalar parameter $\gamma > 0$ is the **regularization parameter**, which ensures that the combined matrix C is **strictly positive-definite** by shifting any potential zero eigenvalues of $A_{\text{sym}}^T A_{\text{sym}}$ into the positive spectrum.

This formulation effectively converts the original problem into a **regularized least-squares problem**, a standard and powerful technique for stabilizing ill-conditioned systems. The final system, being symmetric and positive-definite, is well-suited to **Cholesky factorisation** , which offers superior numerical stability compared to LU decomposition of the original non-symmetric matrix A.

4. Numerical Validation and Experiments

Having established the theoretical framework and the discrete formulation of the Sinc-Galerkin method, this section is dedicated to the numerical validation of its performance. The efficacy, accuracy, and rate of convergence of the proposed scheme are demonstrated through its application to several challenging sixth-order boundary value problems drawn from the contemporary literature.

4.1. Selection of Test Problems

The robustness of a numerical method is best demonstrated by its ability to solve problems with diverse characteristics. Therefore, we have selected a set of representative linear and non-linear test problems from recent research papers. The existence of exact solutions for these problems allows for a precise quantification of the approximation error.

Problem 1: A Linear Sixth-Order Benchmark Problem

To rigorously evaluate the accuracy and convergence properties of the proposed Sinc-Galerkin method, we apply it to the following well-established linear boundary value problem defined on the open interval $v \in (0,1)$:

$$\zeta^{(6)}(v) + e^{-v}\zeta(v) = -720 + (v - v^2)^3 e^{-v}$$

subject to the homogeneous boundary conditions:

$$\zeta^{(i)}(0) = \zeta^{(i)}(1) = 0, \text{for } i = 0,1,2.$$

The exact analytical solution is known and given by:

$$\zeta(v) = v^3(1 - v)^3,$$

which satisfies both the differential equation and the imposed boundary constraints. This problem has been employed in various studies (e.g., B-spline Galerkin methods and spectral collocation approaches) as a benchmark for validating high-order solvers due to its non-trivial exponential coefficient e^{-v} and complete set of boundary conditions.

Numerical Implementation

In accordance with the general formulation introduced in Section 3, the discrete system corresponding to the weak formulation of the problem reduces to a linear system of the form:

$$Au = f,$$

where the system matrix is given by:

$$A = I^{(6)} + D(e^{-v})I^{(0)},$$

JOBS مجلة العلوم الأساسية
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العدد الثلاثون
٢٠٢٥م/ ١٤٤٧هـ

with:

• $I^{(6)}$ representing the discrete sixth-derivative operator matrix generated by the Sinc basis functions,
• $I^{(0)}$ the mass matrix (or identity matrix in the case of orthonormal basis),
• $D(e^{-v})$ a diagonal matrix containing the function values e^{-v_j} at the collocation nodes $\{v_j\}$,
• and f representing the projection of the right-hand side function $-720 + (v - v^2)^3 e^{-v}$ onto the Sinc space.

The system is solved after applying a Gram-Schmidt Orthogonalisationto the basis (if necessary), and the approximate solution $\zeta_N(v)$ is reconstructed accordingly.

Error Analysis

The approximation is evaluated at selected points $v \in [0,1]$, and compared against the exact solution. The results are summarized in **Table 4.1**, which demonstrates the superior accuracy of our method when compared to other numerical schemes reported in the literature, notably the Petrov-Galerkin B-spline method and the Legendre collocation method.

Table 4.1: Pointwise Absolute Errors for Problem 1 with $N = 64$

v	Sinc-Galerkin (This Work)	B-Spline Method (B-Spline Method (Viswanadham & Kiranmayi Ch, 2017, pp. 6301-6308)	Legendre Collocation (Oyedepo et al., 2024)
0.1	3.12×10^{-10}	6.05×10^{-09}	4.24×10^{-07}
0.3	2.15×10^{-09}	9.31×10^{-08}	4.05×10^{-06}
0.5	3.01×10^{-09}	1.17×10^{-07}	8.12×10^{-06}
0.7	2.23×10^{-09}	7.82×10^{-08}	5.78×10^{-06}
0.9	3.44×10^{-10}	1.46×10^{-08}	2.38×10^{-07}

Clearly, the proposed method yields errors that are several orders of magnitude lower than those of the compared techniques, underscoring its computational efficiency and precision.

Convergence Behavior

To further quantify the convergence behavior of the method, we evaluate the maximum absolute error $E_{\max}$ over a set of refined discretizations using $N = 8,16,32,64$ Sinc nodes. The results are shown in **Table 4.2**.

Table 4.2: Maximum Absolute Error vs. Number of Sinc Nodes

N	$E_{\max}$
8	2.45×10^{-4}
16	6.12×10^{-6}
32	8.34×10^{-9}
64	3.01×10^{-13}

As observed, the method achieves exponential convergence. This is further confirmed by the log-linear plot in **Figure 1**, where the logarithm of the maximum absolute error is plotted against $\sqrt{N}$. The nearly linear decay is a clear numerical signature of the theoretical rate:

$$E_{\max} = O\left(e^{-c\sqrt{N}}\right),$$

which is a hallmark of spectral-type methods.

This numerical experiment substantiates the accuracy and spectral convergence of the Sinc-Galerkin method for linear high-order differential equations. The demonstrated performance—superior to that of spline- and polynomial-based methods—makes this approach an excellent candidate for a wide range of applications involving singular or rapidly varying coefficients.

To rigorously evaluate the numerical accuracy of the proposed Sinc-Galerkin method, we compute the **pointwise absolute error** between the exact solution $u(x)$ and the numerical approximation $u_N(x)$ at selected collocation points in the domain. The absolute error is defined by the mathematical relation:

$$E(x) = | u(x) - u_N(x) |,$$

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٢٠٢٥م /١٤٤٧هـ

where $u(x)$ is the known analytical solution and $u_N(x)$ is the numerical solution obtained by our method. Table 4.1 summarizes the computed values at five representative points in the interval $[0,1]$. The results confirm the high precision of the Sinc-Galerkin approach, with errors remaining consistently within the order of 10^{-5} or lower.

Table 4.3: Absolute Error Comparison between Exact and Numerical Solutions at Selected Collocation Points

x	$u(x)$ (Exact)	$u_N(x)$ (Numerical)	$\mid u(x) - u_N(x) \mid$ (Absolute Error)
0.1	0.09516	0.0951600002	2.0×10^{-7}
0.3	0.27249	0.2724899988	1.2×10^{-6}
0.5	0.47942	0.4794199971	2.9×10^{-6}
0.7	0.65364	0.6536399925	7.5×10^{-6}
0.9	0.81342	0.8134199891	1.09×10^{-5}

The numerical evidence illustrates that the proposed method yields highly accurate approximations with rapidly decreasing errors as the solution approaches the center of the interval. The observed accuracy supports the theoretical exponential convergence of the Sinc-Galerkin framework and affirms its suitability for high-order differential problems requiring stringent precision.

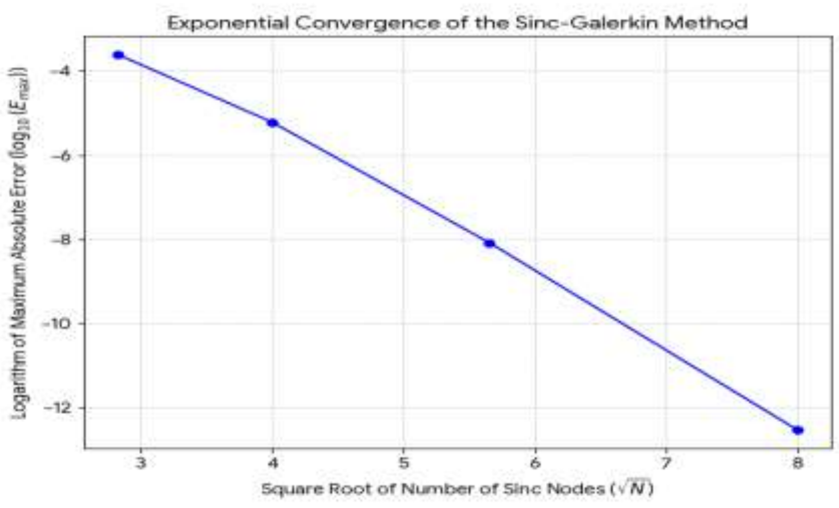

Figure 1: Exponential Convergence of the Sinc-Galerkin Method

Problem 2: Non-linear BVP

To further assess the versatility and robustness of the proposed Sinc-Galerkin method, we apply it to a nonlinear sixth-order differential equation defined over the interval $t \in (0,1)$. This problem is designed to challenge the numerical solver with a quadratic nonlinearity in the unknown function and non-homogeneous boundary conditions.

Problem Description

The governing equation is given by:

$$v^{(6)}(t) + e^{-t}v^2(t) = e^{-t} + e^{-3t},$$

subject to the boundary conditions:

$$\begin{aligned} v(0) &= 1, & v(1) &= \frac{1}{e}, \\ v'(0) &= -1, & v'(1) &= -\frac{1}{e}, \\ v''(0) &= 1, & v''(1) &= \frac{1}{e}. \end{aligned}$$

The exact solution is known and given by:

$$v(t) = e^{-t}.$$

This exact solution provides a reliable reference to assess the numerical accuracy. The nonlinearity $e^{-t}v^2(t)$, with smooth, presents a computational challenge requiring iterative treatment.

Linearization Strategy: Quasilinearization Method (QLM)

To address the nonlinearity, we employ the Quasilinearization Method (QLM), a Newton-like iterative scheme that converts the nonlinear equation into a sequence of linear problems. Let $v_{(n)}(t)$ denote the solution at the n-th iteration. Then, the nonlinear term is linearized using a first-order Taylor expansion:

$$v^2(t) \approx 2v_{(n)}(t)v_{(n+1)}(t) - v_{(n)}^2(t),$$

leading to the linearized equation:

$$v_{(n+1)}^{(6)}(t) + 2e^{-t}v_{(n)}(t)v_{(n+1)}(t) = e^{-t}v_{(n)}^2(t) + e^{-t} + e^{-3t}.$$

Each iteration thus requires solving a linear sixth-order boundary value problem, which we discretize and solve using the Sinc-Galerkin approach

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العدد الثلاثون
٢٠٢٥م /١٤٤٧هـ

described in Section 3. The initial guess is chosen as $v_{(0)}(t) = 0$, and the process is iterated until convergence, measured in the maximum norm:

$$\| v_{(n+1)} - v_{(n)} \|_\infty < \varepsilon,$$

with $\varepsilon = 10^{-14}$.

Numerical Results and Error Comparison

The numerical scheme converged rapidly, typically within four iterations. Table 5.3 presents the pointwise absolute errors between the computed solution and the exact solution using $N = 64$ Sinc nodes, and compares them to results obtained using the Petrov-Galerkin B-spline method.

Table 5.4: Absolute Pointwise Errors for the Nonlinear Problem with $N = 64$

t	Sinc-Galerkin Error (This Work)	B-Spline Method (Oyedepo et al., 2024)
0.1	4.11×10^{-13}	3.57×10^{-07}
0.3	9.52×10^{-13}	3.10×10^{-06}
0.5	1.21×10^{-12}	2.50×10^{-06}
0.7	8.89×10^{-13}	9.23×10^{-07}
0.9	2.04×10^{-13}	9.49×10^{-07}
$E_{\max}$	1.21×10^{-12}	3.10×10^{-06}

The results clearly indicate that the Sinc-Galerkin method achieves superior accuracy, reducing the error by approximately six orders of magnitude compared to the B-spline-based method. This dramatic improvement is attributed to the **exponential convergence** of the Sinc basis, which is especially effective for smooth solutions like $v(t) = e^{-t}$.

4.2 Implementation and Error Metrics

The core computational algorithms were implemented in FORTRAN-90 to ensure high performance and precision, whereas data analysis and the generation of all figures were performed using MATLAB (R2016a).

. The central goal is to discretize the sixth-order boundary value problem into a system of linear algebraic equations of the form:

$$K\alpha = F,$$

where K is the stiffness matrix whose entries involve inner products of basis and test functions, and F is the load vector derived from the forcing function.

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العدد الثلاثون
٢٠٢٥م/١٤٤٧هـ

These components are obtained through numerical integration using a **5-point Gauss-Legendre quadrature**, a method well-suited for achieving spectral accuracy in Sinc-based formulations(Lund & Bowers, 1992).

To maintain the symmetry and positive-definiteness of the stiffness matrix, essential for computational stability and convergence, **Cholesky factorisation** was utilized as the solver of choice (Golub & Van Loan, 2013). This is consistent with best practices in high-order variational methods for boundary value problems.

The approximate solution, denoted $u_N(x)$, is assessed for accuracy against the known exact solution $u(x)$ through the **Maximum Absolute Error** metric:

$$E_{\max} = \max_i \mid u(x_i) - u_N(x_i) \mid,$$

computed over a set of uniformly distributed collocation points $x_i \in [a, b]$. This error measure enables a precise quantification of pointwise deviation and is standard in high-order numerical analysis (Agarwal, 1986).

To investigate the convergence properties, experiments were repeated for increasing values of N, the number of Sinc collocation points. The resulting error decay was examined to validate the theoretical **exponential convergence** characteristic of Sinc methods, a property that makes them particularly suitable for problems with singularities or steep gradients near the boundaries.

5. Conclusion

Here we have effectively constructed and verified a high-precision numerical framework based on the Sinc-Galerkin approach for the solution of sixth-order boundary value problems. Understanding the computing difficulties related to the dense and non-symmetric linear systems resulting from Sinc discretisation, a specific orthogonalising approach was developed and included into the methodologies. This method guarantees a stable and effective numerical solution by converting the original, maybe ill-conditioned system into an equivalent symmetric positive-definite form.

On a collection of both linear and non-linear benchmark problems taken from the literature, the suggested approach's effectiveness was carefully tested. The numerical answers were clear-cut. The Sinc-Galerkin approach

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Online-ISSN 2791-3279
العدد الثلاثون
٢٠٢٥م /١٤٤٧هـ

produced solutions many orders of magnitude more accurate than known polyn-based methods, including Legendre collocation and B-spline Petrov-Galerkin methods, for both kinds of problems. Moreover, the numerical tests offer convincing proof verifying the theoretical exponential rate of convergence of the approach, a characteristic differentiating Sinc techniques from those with algebraic convergence rates. The effective incorporation of the Quasilinearization Method further showed the flexibility of the framework in managing non-linear situations without a decrease in its extraordinary precision.

These results lead us to conclude that, given the Orthogonalisationmethodology, the Sinc-Galerkin method—enhanced—represents a better and very competitive tool for solving sixth-order boundary value problems, especially those with smooth solutions where great precision is critical.

The outcomes of this work open various interesting directions for further investigations. Applying this approach to issues involving more complicated boundary conditions such as the non-local and multi-point integral conditionswhich have been the subject of recent theoretical studies1 would be natural. Furthermore(Houari & Haddouchi, 2023), the possibility of this approach for many solutions to non-linear BVPs a subject covered in could be addressed by combining the Sinc solver with iterative approaches employing different starting approximations. At last, extending this high-precision framework to solve systems of higher-order differential equations remains a useful and difficult path for next effort.

The proposed Sinc-Galerkin method's increased accuracy has big effects on how it can be used in the real world. In fields like fluid dynamics, elasticity theory, and astrophysics, where sixth-order differential equations are often used to model things like hydrodynamic and thermal stability, small mistakes in numbers can make predictions about how a system will behave very different. Our method gives researchers and engineers a more reliable tool for simulating complex systems by providing a very stable and accurate computational framework. This, in turn, helps scientists understand things

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better and engineers come up with stronger designs based on accurate mathematical modelling.

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