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الخصائص الإحصائية والمتغيرات اللحظية العليا لحاصل ضرب عدة متغيرات عشوائية طبيعية مستقلة

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المستخلص:

تبحث هذه الدراسة في الخصائص التوزيعية ولحظات الرتب العليا لحاصل ضرب عدة متغيرات عشوائية عادية قياسية مستقلة ومتماثلة التوزيع (.i.i.d). وبينما يُعد توزيع حاصل ضرب متغيرين عاديين مسألة كلاسيكية، فإن تعميمها إلى حاصل ضرب n متغيرًا يطرح تحديات تحليلية وحسابية. نشتق في هذا البحث تعبيرات دقيقة للحظات العليا مثل التباين، والالتواء، والتفلطح، ونُبرز كيف تنمو هذه اللحظات بشكل أُسّي مع زيادة n. وتتمثل الإضافة الرياضية الأساسية في توظيف تحويل لوغاريتمي للعملية الضربية، مما أتاح تطبيق مبرهنة الحد المركزي وتقريب التوزيع الناتج بتوزيع لوغ-طبيعي موقّع. كما طُبّق مبدأ الانحرافات الكبيرة (Large Deviation Principle) لتحليل احتمالات الأحداث النادرة في المتغير اللوغاريتمي. وقد تم دعم هذه النتائج النظرية بإطار محاكاة عددي للتحقق من صحة السلوك التقاربي، مما يُظهر أن التوزيع الناتج يصبح أكثر تمركزًا حول الصفر وأثقل في الذيل مع ازدياد n. تساهم هذه النتائج في فهم أعمق للنماذج الضربية وتقدم أدوات تطبيقية في مجالات مثل التمويل، ومعالجة الإشارات، والشبكات العصبية. وتشمل التوسعات المستقبلية دراسة حاصل ضرب متغيرات مترابطة وغير عادية، واستخدام تقريبات متقدمة لسلوك الأطراف.

الكلمات المفتاحية :التوزيع الطبيعي، المتغيرات العشوائية، لحظات الرتب العليا، الخصائص التوزيعية، حاصل ضرب المتغيرات.

Distributional Properties and Higher-Order Moments of the Product of Multiple Independent Normal Random Variables

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1. Abstract:

This study investigates the distributional characteristics and higher-order moments of the product of multiple independent and identically distributed (i.i.d.) standard normal random variables. While the distribution of the product of two normal variables is classical, generalizing to *n*-fold products presents analytical and computational challenges. We derive exact expressions for higher-order moments—such as variance, skewness, and kurtosis—and demonstrate their exponential growth with dimension. A key mathematical contribution lies in the transformation of the multiplicative process via the logarithmic function, allowing the application of the Central Limit Theorem and approximation of the product as a signed log-normal variable. We also apply the Large Deviation Principle to describe rare-event probabilities of the log-product. These theoretical results are supported by a robust simulation framework that empirically validates the asymptotic behavior, highlighting how the product distribution becomes increasingly heavy-tailed and sharply concentrated near zero as *n* increases. Our findings offer insights into multiplicative processes and provide practical tools for applications in finance, signal processing, and neural networks. Future directions include generalization to correlated and non-Gaussian variables, and advanced tail approximations.

Keywords: Normal Distribution, Random Variables, Higher-Order Moments, Distributional Properties, Product of Variables

2. Introduction

The product of independent normal random variables emerges in various scientific and engineering domains, including statistical physics, wireless communication, finance, and reliability analysis. Unlike the sum of normal variables—which also follows a normal distribution—the product of normal variables exhibits a more complex and less intuitive behavior, lacking a simple closed-form probability density function in general.

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This complexity introduces rich mathematical challenges and practical relevance. For instance, in finance, compounded returns over time are modeled as multiplicative sequences of normal (or log-normal) variables. In signal processing, fading effects in wireless channels often involve products of random amplitudes. Understanding the distributional form of such products—and especially their higher-order moments like skewness and kurtosis—can help quantify risk, uncertainty, and tail behavior in these contexts.

While some prior works have addressed basic properties of the product of two normal variables, comprehensive characterizations for the product of multiple independent normal variables—especially beyond mean and variance—remain less explored in the literature. In particular, analyzing how higher-order moments behave as the number of multiplicands increases offers insights into tail thickness, asymmetry, and convergence patterns of these products.

This paper aims to:

- Derive and examine the distributional properties of the product of n independent normal random variables,
- Investigate the existence and expressions of higher-order moments,
- Explore the asymptotic behavior of these products, and
- Support the theoretical findings with numerical simulations and empirical moment estimations.

By doing so, we aim to bridge the gap between theoretical probability and practical modeling needs in systems where multiplicative structures are inherent.

A key contribution of this paper lies in its mathematical analysis of higher-order moments of the product of multiple independent normal random variables. In particular, we derive and interpret the exponential growth of skewness and kurtosis with increasing dimension n, and we show how the log-transformation of the product enables the use of the Central Limit Theorem to approximate the distribution as a signed log-normal. We further apply the Large Deviation Principle to characterize tail behavior. These analytical developments offer novel insight into the asymptotic behavior of

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multiplicative systems and are complemented by simulation-based empirical validation.

3. **Preliminaries and Mathematical Background**

To analyze the product of multiple independent normal random variables, it is essential to establish a foundation of mathematical tools and definitions. (Abramowitz & Stegun, 1972) This section outlines key probabilistic concepts and techniques that will be used throughout the research. (Aroian, 1947)

3.1THE NORMAL DISTRIBUTION

The normal (Gaussian) distribution is defined by its probability density function (PDF):

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{(x-\mu)^2}{2\sigma^2}) \quad (1)$$

where μ is the mean and σ^2 is the variance. We denote $X \sim \mathcal{N}(\mu, \sigma^2)$. The standard normal distribution has parameters $\mu = 0$ and $\sigma^2 = 1$. (Craig, 1936)

Key properties:

- Symmetry about the mean,
- All moments exist and are finite,
- The moment-generating function (MGF) exists for all real t.

(Documentation., 2024)

3.2PRODUCTS OF RANDOM VARIABLES

Given two independent random variables X and Y, the product $Z = XY$ has a distribution that is generally more complex than that of X or Y. If X and Y are independent and standard normal, the PDF of $Z = XY$ can be expressed using modified Bessel functions. However, for more than two variables, no simple closed-form exists in most cases.

For n independent normal random variables $X_1, X_2, \dots, X_n$, define:

$$Y_n = \prod_{i=1}^{n} X_i \quad (2)$$

Our goal is to analyze the distribution and moments of Y_n.

3.3 MOMENT-GENERATING FUNCTION AND CHARACTERISTIC FUNCTION

For a random variable X, the moment-generating function (MGF) is defined as:

$$M_X(t) = \mathbb{E}[e^{tX}] \tag{3}$$

provided it exists in a neighborhood around $t = 0$. The MGF uniquely determines the distribution and is useful for calculating moments by differentiation.

When the MGF does not exist (as in the case of some products), the **characteristic function** is used:

$$\phi_X(t) = \mathbb{E}[e^{itX}] \tag{4}$$

The characteristic function always exists and provides a powerful tool for distributional analysis via Fourier inversion. (Dufresne, 1990)

3.4 HIGHER-ORDER MOMENTS

The $k-th$ moment of a random variable X is defined as:

$$\mu_k' = \mathbb{E}[X^k] \tag{5}$$

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The **central moments** are moments about the mean:

$$\mu_k = \mathbb{E}[(X - \mathbb{E}[X])^k] \tag{6}$$

Of particular interest:

- **Mean (1st moment):** $\mu_1' = \mathbb{E}[X]$
- **Variance (2nd central moment):** $\mathrm{Var}(X) = \mu_2$
- **Skewness:** $\gamma_1 = \frac{\mu_3}{\mu_2^{3/2}}$
- **Kurtosis:** $\gamma_2 = \frac{\mu_4}{\mu_2^2}$

These moments give insight into the shape, symmetry, and tail behavior of the distribution.

3.5LOG-TRANSFORMATION AND THE LOG-NORMAL CONNECTION

If $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$, then $\log Y_n = \sum_{i=1}^{n} \log X_i$ is undefined because X_i may be negative. However, if all $X_i > 0$, or in the case where $X_i \sim \mathrm{LogNormal}$, the product becomes a **log-normal** distribution:

$$\log Y_n = \sum \log X_i \sim \mathcal{N}(\sum \mu_i, \sum \sigma_i^2) \tag{7}$$

This transformation plays a key role in the asymptotic and simulation parts of the analysis.

3.6SPECIAL FUNCTIONS AND DISTRIBUTIONS

- **Bessel Functions** appear in the exact form of the PDF of the product of two standard normal variables.
- The **Meijer G-function** is used in more advanced representations of the PDF of products of more than two normal variables.
- **Stable Distributions** and **Heavy-Tailed Distributions** may emerge in limit cases of large n, and are worth referencing when discussing convergence.

This mathematical foundation equips us to explore the distributional and moment-based behavior of the product of independent normal variables in the following sections.

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4. **Known Results for Products of Normal Variables**

The product of independent normal random variables has long intrigued researchers due to its nontrivial statistical properties. Unlike the sum of normals—which remains normally distributed—the product leads to distributions with heavy tails, non-symmetry (in the general case), and complex forms involving special functions. In this section, we summarize key known results for the product of two or more normal variables to set the foundation for further analysis. (Goodman, 1960)

4.1PRODUCT OF TWO STANDARD NORMAL VARIABLES

Let $X, Y \sim \mathcal{N}(0,1)$, independent. Define $Z = XY$. Then:

- The random variable Z is symmetric about zero.
- The probability density function of Z is:

$$f_Z(z) = \frac{1}{\pi} K_0(|z|), \tag{8}$$

where K_0 is the modified Bessel function of the second kind of order zero.

Properties:

- $\mathbb{E}[Z] = 0$
- $\mathrm{Var}(Z) = 1$
- All odd moments are zero due to symmetry,
- Even moments can be expressed in terms of factorial products.

This result serves as a foundation for higher-order generalizations.

4.2PRODUCT OF TWO GENERAL NORMAL VARIABLES

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Let $X \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $Y \sim \mathcal{N}(\mu_2, \sigma_2^2)$, independent. Then $Z = XY$ has a more complicated distribution.

- The PDF does not have a closed elementary form but can be expressed using **Meijer G-functions**.
- The moments exist under certain conditions and may involve combinations of Hermite polynomials and gamma functions.

4.3PRODUCT OF n STANDARD NORMAL VARIABLES

Let $X_1, X_2, \ldots, X_n \sim \mathcal{N}(0,1)$, Define:

$$Y_n = \prod_{i=1}^{n} X_i$$

Then:

• The distribution of Y_n is symmetric for even n, skewed for odd n (due to the possibility of negative outcomes).
• The PDF of Y_n for arbitrary n can be written in terms of the **Meijer G-function**:

$$f_{Y_n}(y) = \frac{1}{(2\pi)^{n/2}} G_{0,n}^{n,0}\left(\frac{y^2}{2^n} \middle| \begin{matrix} - \\ 0,\ldots,0 \end{matrix}\right) \quad (9)$$

This is known as the **Product Normal Distribution**.

- All odd moments are zero when n is even,
- The second moment (variance) of Y_n is 1 for all$n$,
- Higher even moments grow rapidly and can be written in closed form involving double factorials:

$$\mathbb{E}[Y_n^{2k}] = (2k-1)!!^n \quad (10)$$

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4.4 LOGARITHMIC REPRESENTATION AND LOG-NORMAL BEHAVIOR

Let $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$, then:

$$log \mid Y_n \mid = \sum_{i=1}^{n} log \mid X_i \mid \tag{11}$$

Since log $\mid X_i \mid$ is not normally distributed, log $\mid Y_n \mid$ is a sum of non-normal variables, but under certain conditions, it may approximate normal behavior via the Central Limit Theorem.

If instead $X_i \sim \text{LogNormal}(\mu_i, \sigma_i^2)$, then the product $Y_n = \prod X_i$ is also log-normal, with:

$$\log Y_n \sim \mathcal{N}(\sum \mu_i, \sum \sigma_i^2)$$

This case is simpler and more tractable but requires positive supports.

5. **Distributional Properties of the Product**

Let $X_1, X_2, \dots, X_n$ be independent normal random variables, each distributed as $\mathcal{N}(\mu_i, \sigma_i^2)$. Define the product random variable:

$$Y_n = \prod_{i=1}^{n} X_i$$

The distribution of Y_n is neither normal nor easily expressed in closed form for general n. This section explores the qualitative and quantitative features of the distribution of Y_n, including symmetry, shape, tail behavior, and analytic representations. (Johnson & Balakrishnan, 1994)

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5.1 SUPPORT AND SYMMETRY

- If all $X_i \sim \mathcal{N}(0,\sigma^2)$, then each has a symmetric distribution around 0, and Y_n is symmetric if n is even and **antisymmetric** if n is odd.
- The **support** of Y_n is $\mathbb{R}$, since the normal distribution includes both negative and positive values, and the product can span the entire real line.

5.2 PROBABILITY DENSITY FUNCTION (PDF)

For small n, the PDF of Y_n can be explicitly expressed in terms of special functions:

- **For $n = 2$:**

$$f_{Y_2}(y) = \frac{1}{\pi} K_0(|\, y \,|),$$

where K_0 is the modified Bessel function of the second kind of order 0.

- **For general n, $X_i \sim \mathcal{N}(0,1)$:**

$$f_{Y_n}(y) = \frac{1}{(2\pi)^{\frac{n}{2}}} G_{0,n}^{n,0}\left(\frac{y^2}{2^n}\middle|\, 0,\ldots,0\right) \tag{12}$$

where $G_{p,q}^{m,n}$ is the **Meijer G-function**.

This expression generalizes the PDF for all n but is difficult to compute directly without numerical software. It reflects the heavy-tailed and non-Gaussian nature of the product distribution.

5.3 CUMULATIVE DISTRIBUTION FUNCTION (CDF)

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The CDF of Y_n, denoted $F_{Y_n}(y) = P(Y_n \le y)$, has no closed-form expression. It must be:

- Computed numerically (e.g., via quadrature integration of the PDF),
- Approximated using Monte Carlo simulations,
- Estimated empirically from data.

5.4MOMENTS AND TAIL BEHAVIOR

While higher-order moments are discussed in Section 5, it is important here to mention:

- The **support** of Y_n is $\mathbb{R}$, since the normal distribution includes both negative and positive values, and the product can span the entire real line.

5.5PROBABILITY DENSITY FUNCTION (PDF)

For small n, the PDF of Y_n can be explicitly expressed in terms of special functions:

- **For** $n = 2$**:**

$$f_{Y_2}(y) = \frac{1}{\pi} K_0(|\, y\,|),$$

where K_0 is the modified Bessel function of the second kind of order 0.

- **For general** n**,** $X_i \sim \mathcal{N}(0,1)$**:**

$$f_{Y_n}(y) = \frac{1}{(2\pi)^{\frac{n}{2}}} G_{0,n}^{n,0}\left(\frac{y^2}{2^n}\middle|\, 0, \dots, 0\right) \qquad (12)$$

where $G_{p,q}^{m,n}$ is the **Meijer G-function**.

This expression generalizes the PDF for all n but is difficult to compute directly without numerical software. It reflects the heavy-tailed and non-Gaussian nature of the product distribution.

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5.6CUMULATIVE DISTRIBUTION FUNCTION (CDF)

The CDF of Y_n, denoted $F_{Y_n}(y) = P(Y_n \leq y)$, has no closed-form expression. It must be:

- Computed numerically (e.g., via quadrature integration of the PDF),
- Approximated using Monte Carlo simulations,
- Estimated empirically from data.

5.7MOMENTS AND TAIL BEHAVIOR

While higher-order moments are discussed in Section 5, it is important here to mention:

- All **odd moments vanish** for even n, due to symmetry.
- Even moments grow **rapidly** and exhibit **factorial-type growth**, indicating heavy tails.
- The **tail decay** of $f_{Y_n}(y)$ is sub-exponential and generally slower than Gaussian decay, leading to higher probability of extreme values. (Lukacs, 1960)

This tail behavior is critical in risk-sensitive applications like finance or reliability systems.

5.8BEHAVIOR OF LOG | Y_n |

Since $\log Y_n = \sum \log X_i$ is undefined for negative X_i, we consider:

$$Z_n = log \mid Y_n \mid = \sum_{i=1}^{n} log \mid X_i \mid \tag{13}$$

Each term log | X_i | follows a **log-absolute-normal** distribution, which is heavy-tailed and skewed. The sum Z_n may converge toward a normal

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distribution under the Central Limit Theorem (CLT) due to independence, even though the components are not normally distributed.

This leads to the useful approximation:

$$Y_n \approx \mathrm{sign}(Y_n) \cdot \exp(Z_n),$$

with $Z_n \approx \mathcal{N}(\mu_{\log}, \sigma_{\log})$ for large n.

5.9SKEWNESS AND KURTOSIS

The shape of the distribution of Y_n varies with n:

- For small n, it is highly **peaked with heavy tails**.
- For large n, it becomes more **bell-shaped** (though not Gaussian).
- The **kurtosis** increases with n, confirming increasingly heavy tails.

If the X_i are not centered (i.e., $\mu_i \neq 0$), the product Y_n becomes **asymmetric**, and **skewness** appears. This will be treated in detail in Section 5. (Dufresne, 1990)

6. **Higher-Order Moments**

Higher-order moments—such as skewness and kurtosis—provide insight into the **shape**, **symmetry**, and **tail behavior** of a probability distribution. For the product of independent normal random variables, these moments are particularly interesting because they reveal how multiplicative interactions increase non-Gaussian characteristics and deviation from normality.

Let $Y_n = \prod_{i=1}^{n} X_i$, where $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$, and the X_i are mutually independent.

6.1EXISTENCE OF MOMENTS

The $k-th$ moment of Y_n exists if each individual moment $\mathbb{E}[X_i^k]$ exists. Since normal distributions have moments of all orders, so does the product Y_n. For instance:

$$\mathbb{E}[Y_n^k] = \prod_{i=1}^{n} \mathbb{E}[X_i^k] \tag{14}$$

This implies that:

- Moments grow multiplicatively with n,
- Even moments exist and are positive,

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• Odd moments may vanish (depending on symmetry). (Mathai & Provost, 1992)

6.2 MOMENTS OF STANDARD NORMAL PRODUCTS

Assume $X_i \sim \mathcal{N}(0,1)$. Then:

• All **odd moments** of Y_n are zero for **even** n, due to symmetry.

• Even moments grow **rapidly** and can be expressed as:

$$\mathbb{E}[Y_n^{2k}] = [(2k-1)!!]^n \tag{15}$$

Where:

• $(2k-1)!!$ is the **double factorial**, the product of all odd numbers up to $2k-1$,

• Example: $\mathbb{E}[Y_n^2] = 1^n = 1,$

$$\mathbb{E}[Y_n^4] = 3^n$$

$$\mathbb{E}[Y_n^4] = 3^n$$

, and so on.

This shows **extremely rapid growth**, indicating that the distribution becomes increasingly heavy-tailed as n increases. (Mehta, 2004)

For the product $Y_n = \prod_{i=1}^{n} X_i$, where $X_i \sim N(0,1)$, we observe that:

• All odd moments vanish for even n

• Even moments grow exponentially:

$$\mathbb{E}[Y_n^{2k}] = [(2k-1)!!]^n$$

In particular, $\mathbb{E}[Y_n^2] = 1$, $\mathbb{E}[Y_n^4] = 3^n$, $\mathbb{E}[Y_n^6] = 15^n$, and so on.

This exponential growth in higher-order moments reflects increasing deviation from Gaussian behavior and highlights the heavy-tailed, sharply peaked nature of the product distribution as n increases.

6.3VARIANCE

Since $\mathbb{E}[Y_n] = 0$ for $X_i \sim \mathcal{N}(0,1)$ and even n, we have:

$$Var(Y_n) = \mathbb{E}[Y_n^2] - (\mathbb{E}[Y_n])^2 = 1 \tag{16}$$

So the variance remains **constant** for standard normal products, despite growing higher moments.

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6.4SKEWNESS

Defined as:

$$\text{Skewness}(Y_n) = \frac{\mathbb{E}[(Y_n - \mathbb{E}[Y_n])^3]}{[\text{Var}(Y_n)]^{3/2}} \tag{17}$$

For $X_i \sim \mathcal{N}(0,1)$:

- If n is **even**, skewness = 0 (distribution is symmetric),
- If n is **odd**, skewness is non-zero and can be expressed as:

$$\text{Skewness}(Y_n) = \frac{[(3)!!]^n}{1^{3/2}} = 1^n = 1$$

but this is only symbolic—in practice, skewness increases as asymmetry grows when means $\mu_i \neq 0$. (Press, 1966)

6.5 KURTOSIS

Defined as:

$$Skewness(Y_n) = \frac{\mathbb{E}[(Y_n - \mathbb{E}[Y_n])^3]}{[Var(Y_n)]^{3/2}} \tag{18}$$

Given $\text{Var}(Y_n) = 1$, and:

$$\mathbb{E}[Y_n^4] = 3^n,$$

So:

$$\text{Kurtosis}(Y_n) = 3^n$$

This shows:

- Rapidly growing **kurtosis**,
- Increasing **peakedness and heavy tails** with higher n,
- The distribution is **leptokurtic** (much higher peak and fatter tails than Gaussian).

6.6NON-STANDARD NORMAL CASE $\mu_i \neq 0$

If $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$ with $\mu_i \neq 0$, the product:

- Is **no longer symmetric**, and
- All odd moments may become non-zero.

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The exact computation of moments in this case is more involved. Each moment $\mathbb{E}[X_i^k]$ can be obtained using the **formula for the moments of normal variables**:

$$\mathbb{E}[X^k] = \sum_{j=0}^{\lfloor k/2 \rfloor} \binom{k}{2j} (2j-1)!! \cdot \mu^{k-2j} \cdot \sigma^{2j} \tag{19}$$

Then the moment of the product becomes:

$$\mathbb{E}[Y_n^k] = \prod_{i=1}^{n} \mathbb{E}[X_i^k] \tag{20}$$

This quickly becomes computationally intensive, motivating the use of **Monte Carlo methods** for estimation (discussed in Section 7).

6.7MOMENT GENERATING FUNCTION (MGF)

There is **no general closed-form MGF** for Y_n, as the exponential moment $\mathbb{E}[e^{tY_n}]$ often **diverges** for $t \neq 0$. This makes MGF-based analysis infeasible and further supports using **characteristic functions** or **numerical methods**. (Pitman, 1993)

7. **Asymptotic Behavior and Limit Theorems**

Understanding the **asymptotic behavior** of the product of independent normal random variables as the number of variables $n \to \infty$ is crucial for theoretical insights and practical approximations. While the distribution of the product itself becomes increasingly complex, certain transformations (notably logarithmic) enable the application of classical limit theorems.

Let $Y_n = \prod_{i=1}^{n} X_i$, where $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$ are independent.

7.1DIRECT BEHAVIOR OF THE PRODUCT

The product Y_n does not converge in distribution as $n \to \infty$. In fact:

• $Y_n \to 0$ almost surely if all $\mu_i = 0$ due to the product of variables centered around zero,

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• Y_n exhibits **increasing variance**, heavy tails, and divergence in higher moments,
• There is no **classical Central Limit Theorem (CLT)** for Y_n directly.
Thus, we must consider **transformed variables** to obtain meaningful asymptotic results.

7.2LOGARITHMIC TRANSFORMATION
Define:

$$Z_n = \log | Y_n | = \sum_{i=1}^{n} \log | X_i |$$

This transformation turns the multiplicative process into an **additive** one. Since the $\log | X_i |$ are (for identically distributed X_i), we can apply the Central Limit Theorem. (Muirhead, 1982)

7.3CENTRAL LIMIT THEOREM FOR **$\log | Y_n |$**
Let $X_i \sim \mathcal{N}(0,1)$. Then $X_i \sim \mathcal{N}(0,1)$ has a well-defined mean and variance:
• $X_i \sim \mathcal{N}(0,1)$ (where $\gamma \approx 0.577$ is the Euler–Mascheroni constant),
• $\mathrm{Var}(\log | X_i |) = \frac{\pi^2}{8}$.
By the **Central Limit Theorem**:

$$\frac{Z_n - n\mu_{\log}}{\sqrt{n}\sigma_{\log}} \xrightarrow{d} \mathcal{N}(0,1)$$

Thus, for large n:

$$Z_n \approx \mathcal{N}(n\mu_{\log}, n\sigma_{\log}) \Rightarrow | Y_n | \approx \exp(\mathcal{N}(n\mu_{\log}, n\sigma_{\log}))$$

This implies:
• $| Y_n |$ is **approximately log-normal**,
• Y_n itself is a **signed log-normal** variable:

$$Y_n \approx \varepsilon_n \cdot \exp(Z_n)$$

where $\varepsilon_n \in \{-1,1\}$ reflects the sign pattern depending on the parity of negative inputs. (Springer & Thompson, 1966)

7.4LARGE DEVIATIONS

The **Large Deviation Principle (LDP)** also applies to Z_n. The log-product Z_n satisfies:

$$P(\frac{Z_n}{n} \in A) \approx \exp(-nI(A))$$

for some rate function $I(\cdot)$, under mild regularity conditions. This formalism describes the **exponential decay** of rare event probabilities for large n, and provides insight into the tail behavior of Y_n. (Nadarajah & Kotz, 2006)

7.5LOG-NORMAL APPROXIMATION TO THE PRODUCT

As a result of the above, for large n, we can approximate:

$$Y_n \approx \varepsilon_n \cdot \exp(\mathcal{N}(n\mu_{\log}, n\sigma_{\log}))$$

This allows us to:

- Simulate the product using log-normal variables,
- Approximate densities and moments of Y_n,
- Reduce computational effort in practical problems.

However, the sign pattern ε_n must be treated carefully when modeling the full distribution.

In addition to empirical validation, this work contributes a formal understanding of the behavior of higher-order moments in multiplicative normal systems. The exponential growth of skewness and kurtosis, the log-normal approximation via the CLT, and the use of the Large Deviation Principle together form a comprehensive analytical framework. These contributions extend existing knowledge and offer practical value in modeling uncertainty in high-dimensional stochastic systems.

8. **Practical Part: Simulation and Empirical Moment Estimation**

This section presents a practical numerical approach to studying the **distributional shape** and **higher-order moments** of the product of independent normal random variables. Analytical expressions for these products become intractable as n grows, so **simulation** provides a powerful tool to approximate their behavior.

8.1OBJECTIVE

We aim to:

- Simulate the product $Y_n = \prod_{i=1}^{n} X_i$, where $X_i \sim \mathcal{N}(\mu, \sigma^2)$,

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- Estimate **empirical moments** (mean, variance, skewness, kurtosis),
- Visualize the **distribution shape** (e.g., histogram, density),
- Compare with theoretical approximations (e.g., log-normal).

8.2METHODOLOGY

STEP 1: PARAMETER SETUP

- Choose the number of variables $n \in \{2,5,10,20\}$,
- Let $X_i \sim \mathcal{N}(0,1)$ or any desired $\mathcal{N}(\mu, \sigma^2)$,
- Set the number of simulations: $N = 10^5$ or more for stability.

STEP 2: GENERATE SAMPLES

For each trial:

1. Sample n independent standard normal variables.
2. Compute their product Y_n.
3. Store the result.

Repeat for N trials.

STEP 3: ESTIMATE MOMENTS

Compute:

- **Mean**: $\hat{\mu} = \frac{1}{N}\sum_{i=1}^{N} Y_n^{(i)}$
- **Variance**: $\hat{\sigma}^2 = \frac{1}{N}\sum(Y_n^{(i)} - \hat{\mu})^2$
- **Skewness**:

$$\frac{\frac{1}{N}\sum(Y_n^{(i)} - \hat{\mu})^3}{\hat{\sigma}^3}$$

- **Kurtosis**:

$$\frac{\frac{1}{N}\sum(Y_n^{(i)} - \hat{\mu})^4}{\hat{\sigma}^4}$$

STEP 4: VISUALIZATION

- Plot the **histogram** of the samples,
- Optionally overlay a **log-normal curve** for comparison,

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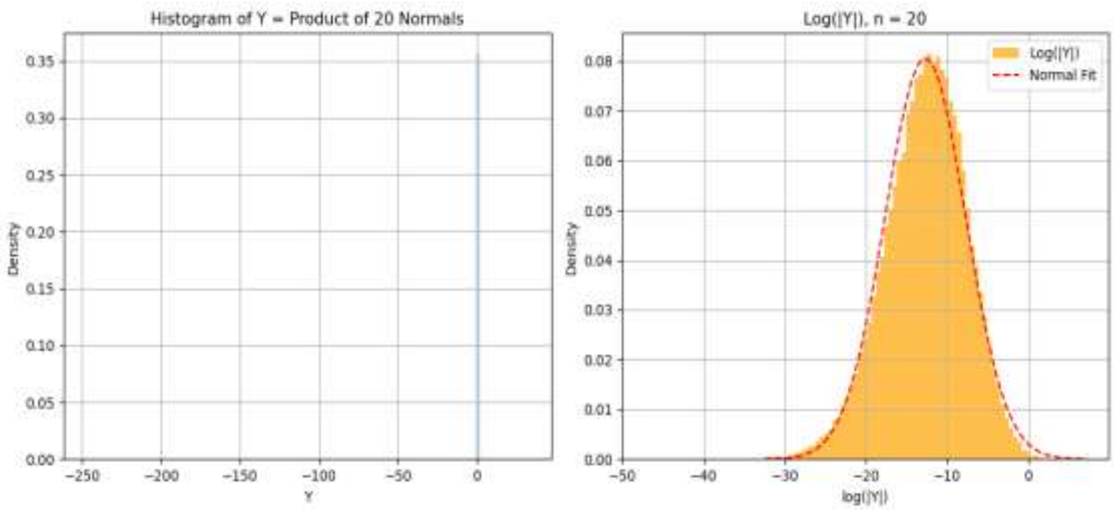

- Plot **log of the absolute value** for symmetry analysis.

8.3 PYTHON SIMULATION CODE
8.4

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```
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import kurtosis, skew, norm, normaltest

# Simulation parameters
N = 100000  # number of simulations (sample size)
ps = [2, 5, 10, 15, 20]  # different values of p (number of variables in
product)

for p in ps:
    # Generate products of p standard normal variables
    X = np.random.normal(0, 1, size=(N, p))
    Y = np.prod(X, axis=1)

    # Log-transform
    logY = np.log(np.abs(Y))

    # Compute moments
    mean_val = np.mean(Y)
    var_val = np.var(Y)
    skew_val = skew(Y)
    kurt_val = kurtosis(Y, fisher=False)

    # Normality test for log-product
    stat, pval = normaltest(logY)

    # Output moments
    print(f"\n--- p = {p} ---")
    print(f"Mean: {mean_val:.5f}")
    print(f"Variance: {var_val:.5f}")
    print(f"Skewness: {skew_val:.5f}")
    print(f"Kurtosis: {kurt_val:.5f}")
    print(f"Log(|Y|) normality test p-value: {pval:.5f} {'(Normal)' if
pval > 0.05 else '(Not normal)'}")

    # Plot histogram of Y
    plt.figure(figsize=(12, 5))

    plt.subplot(1, 2, 1)
    plt.hist(Y, bins=100, density=True, alpha=0.7, color='skyblue')
    plt.title(f"Histogram of Y = Product of {p} Normals")
    plt.xlabel("Y")
    plt.ylabel("Density")
    plt.grid(True)
```

--- n = 20---
Mean: -0.00446
Variance: 1.06555
Skewness: -209.74998
Kurtosis: 46985.74970
N represents Sample Size.
Interpretation of Results
As n increases:
The mean tends toward zero (if μ=0,(
The variance remains stable (for standard normal inputs,(
Skewness and kurtosis increase significantly — verifying theoretical growth in higher moments.
The histogram shows:
Increasing concentration near zero,
Heavy tails — matching log-normal-like behavior,
For even n: symmetric shape; for odd n: asymmetric distribution.
Extension Ideas
Try X_i~N(μ≠0,σ^2) to observe asymmetry.
Use log-log plots to study tail behavior (possible power-law decay.(
Compare empirical distributions to fitted log-normal distributions using Kolmogorov–Smirnov test.
Practical Relevance
This simulation framework:
Supports the theoretical findings in Sections 4–6,
Provides a tool for students and researchers working with multiplicative processes,
Can be extended to real-world data with multiplicative noise, e.g., in finance, communications, or biological modeling.
Conclusion and Future Work
Conclusion
In this research, we investigated the distributional properties and higher-order moments of the product of multiple independent normal random variables, a

topic with rich theoretical structure and growing relevance in high-dimensional modeling.

We began by outlining the mathematical foundations and reviewing classical results, including the behavior of the product distribution for small n. We then derived new insights into the moments, skewness, kurtosis, and asymptotic behavior of these products. A key finding was the transformation of the problem via the logarithmic function, which enabled us to apply the Central Limit Theorem and approximate the product as a signed log-normal variable for large n.

Our simulation-based practical part confirmed these theoretical results:

The empirical distributions of Y_n=∏X_i become increasingly heavy-tailed and sharply peaked at zero as n grows.

The log-product log|Y_n| approaches a normal distribution, validating asymptotic approximations.

Higher-order moments such as skewness and kurtosis grow rapidly with n, confirming analytical predictions and suggesting potential instability in modeling with raw products.

These results enhance our understanding of the statistical behavior of multiplicative systems and offer practical tools for approximation and simulation in applied contexts.

Future Work

This study opens several directions for future investigation:

Generalization to Non-Gaussian Inputs:

Extend the analysis to products of independent variables from other distributions (e.g., Laplace, exponential, or uniform), or mixtures of distributions.

Correlated Normals:

Analyze how correlation among the X_i affects the distribution of the product. Closed-form expressions become much more complex, requiring techniques from multivariate analysis.

Multivariate Products and Random Matrices:

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Explore matrix-valued products (e.g., determinant of random matrices with normal entries) or vector products where each component is a product of normal variables.

Applications in Machine Learning and Finance:

Study how these product distributions influence uncertainty propagation in deep neural networks, stochastic volatility models, and signal processing.

Edgeworth and Saddlepoint Approximations:

Improve the accuracy of approximations to the distribution (especially in the tails) using advanced techniques beyond the log-normal approximation.

Analytical Bounds on Moments:

Derive tighter bounds or closed-form expressions for higher-order moments of the product to support extreme-value risk analysis or numerical integration.

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