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طريقة تشيبيشيف الطيفية لمعادلات الحرارة الكسرية الزمنية على مجالات غير منتظمة

بادي رحيم بهك
مدرس في ثانوية نخب العزيزية للمتفوقين
badyrhymaltmymy@gmail.com

المستخلـــص:

في هذا البحث، نستخدم كثيرات حدود تشيبيشيف لإنشاء طريقة تجميع طيفية تُستخدم لحل معادلة الحرارة الكسرية الزمنية ثنائية الأبعاد عدديًا باستخدام مشتقة كابوتو. وتتميز طريقتنا عن الطرق التقليدية بدمجها الدقة العالية للتفكيك الطيفي الشامل في الفضاء مع تقريب غرونوالد–ليتنيكوف المُزاح في الزمن. يعمل المخطط المقترح على المجالات الحسابية غير المنتظمة من خلال استخدام تحويلات إحداثية لرسم الهندسة على مجال قياسي. نقوم بدراسة تقارب الطريقة، واستقرارها، وكفاءتها الحسابية. كما نختبر الطريقة على عدد من المسائل القياسية في مجالات منتظمة وغير منتظمة، مثل شبه منحرف قائم وشكل أنبوبي. وتُظهر الاختبارات العددية أن الطريقة الطيفية تحقق تقاربًا أسيًا للحلول الناعمة، كما أنها أكثر دقة من طرق التفكيك الأخرى. تم إجراء جميع التجارب العددية باستخدام MATLAB R2023a على نظام Windows 11 مزود بمعالج Intel Core i5 وذاكرة RAM بسعة ١٦ جيجابايت.

الكلمات المفتاحية: معادلة الحرارة الكسرية الزمنية، مشتقة كابوتو، الطريقة الطيفية، تجميع تشيبيشيف، المجالات غير المنتظمة، التحليل العددي.

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٢٠٢٥م /١٤٤٧هـ

A Chebyshev Spectral Method for Time–Fractional Heat Equations on Irregular Domains

Badi Rahim Bahk
Teacher at Nakhbat Al-Aziziyah High School for Gifted Students
badyrhymaltmymy@gmail.com

Abstract:
In this paper, we use Chebyshev polynomials to create a spectral collocation method that can be used to numerically solve the two-dimensional time-fractional heat equation with the Caputo derivative. Our method is different from traditional ones because it uses the high accuracy of global spectral discretization in space along with the shifted Grünwald–Letnikov approximation in time. The suggested scheme works on irregular computational domains by using coordinate transformations to map the geometry onto a standard domain. We look at how the method converges, how stable it is, and how fast it works. We test the scheme on a number of benchmark problems in both standard and irregular domains, such as a right trapezoid and a pipe-like shape. Numerical tests show that the spectral method achieves exponential convergence for smooth solutions, and it is more accurate than other methods of discretization. The numerical experiments were conducted using MATLAB R2023a on a Windows 11 system equipped with an Intel Core i5 processor and 16 GB of RAM.

Keywords: Time–fractional heat equation, Caputo derivative, spectral method,Chebyshev collocation, irregular domains, numerical analysis.

1 Introduction

Fractional differential equations (FDEs) have become important for modeling processes that depend on memory and history in many fields, including finance, heat transfer, viscoelasticity, and anomalous diffusion (Canuto et al., 2006, p. 15; Podlubny, 1999, p. 50). The time-fractional heat equation is one of the most useful for describing sub-diffusive phenomena, which happen when the rate of diffusion is slower than what the classical Fourier law says it should be. There aren't many analytical solutions to fractional heat equations, and when there are, they are usually only for simple shapes or perfect conditions. This has led to the creation of numerical methods that can handle a wide range of situations, even those with uneven shapes and boundary conditions. There are many different numerical methods that have been suggested, including finite difference methods (Mohebbi, Abbaszadeh, & Dehghan, 2013, p. 40), meshless methods (Dehghan, Mohammadi, & Hamidi, 2014, p. 187; Pang, Lu, & Karniadakis, 2019, p. A2605), finite block methods (Li & Wen, 2014, p. 374), and spectral methods (Shen, Tang, & Wang, 2011, p. 10). Spectral methods, especially those that use Chebyshev or Legendre polynomials, are very accurate and don't have a lot of degrees of freedom. This is especially true for problems with smooth solutions. These methods turn differential equations into algebraic systems by using global basis functions. They work well for problems with simple shapes. When used on irregular domains, the spectral accuracy must be kept by making the right coordinate transformations (Canuto et al., 2006, p. 55).

This paper develops a Chebyshev spectral collocation method for a two-dimensional time-fractional heat equation with a Caputo time derivative. Time is discretized by means of a shifted Grünwald–Letnikov approximation (Tian, Zhou, & Deng, 2015, p. 1705), while space is discretized using Chebyshev polynomials defined on collocation nodes. Irregular domain boundaries are converted to a standard square domain for the sake of simplification.

This paper follows a defined structure. In Section 2, we introduce some mathematical preliminaries and definitions needed for our problem. We include the formulation for the spectral method in Section 3. Section 4 describes the fully-discrete scheme. Section 5 features extensive numerical results for multiple domains. Section 6 contains analysis for the error and convergence of the methods presented in the earlier sections. We conclude the paper in Section 7.

2 Preliminaries

In this part we introduce some of the basic concepts and notations that will be used throughout the paper such as Caputo fractional derivative, Chebyshev collocation points, and some spaces of functions related to spectral approximation.

2.1 Caputo Fractional Derivative

Let $0 < \alpha < 1$. The Caputo fractional derivative of order α for a function $u(t) \in C^1[0,T]$ is defined as:

$$ {}^{C}D_t^{\alpha}u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u\prime(\xi)}{(t-\xi)^{\alpha}} d\xi, (1)$$

where $\Gamma(\cdot)$ stands for the Gamma function. This specific form of the derivative is particularly helpful for physical problems because it accommodates initial conditions of integer order (Podlubny, 1999, p. 45).

To perform a time discretization of the Caputo derivative, we apply the shifted Grünwald–Letnikov approximation (Tian, Zhou, & Deng, 2015, p. 1705).

$$ {}^{C}D_t^{\alpha}u(t_n) \approx \sum_{k=0}^{n} w_k^{(\alpha)} u(t_{n-k}) + \mathcal{O}(\tau^2), (2)$$

with weights defined recursively as:

$$l_0^{(\alpha)} = 1, \quad l_k^{(\alpha)} = \left(1 - \frac{\alpha+1}{k}\right) l_{k-1}^{(\alpha)}, \quad k \geq 1, (3)$$

$$w_0^{(\alpha)} = \tau^{-\alpha}\left(1 + \frac{\alpha}{2}\right) l_0^{(\alpha)}, (4)$$

$$w_k^{(\alpha)} = \tau^{-\alpha}\left(\left(1 + \frac{\alpha}{2}\right) l_k^{(\alpha)} - \frac{\alpha}{2} l_{k-1}^{(\alpha)}\right), \quad k \geq 1, (5)$$

where τ is the time step and $t_n = n\tau$.

2.2 Chebyshev Collocation Points

Let N be the number of spatial grid points in one dimension. The Chebyshev-Gauss-Lobatto collocation points on the interval [a, b] are defined by:

$$x_j = \frac{a+b}{2} + \frac{b-a}{2}\cos\left(\frac{j\pi}{N}\right), \quad j = 0,1,\dots,N. (6)$$

These points are distributed non-uniformly, with a clustering near the boundaries a and b. This property significantly alleviates the Runge phenomenon, which typically appears when interpolating functions using equidistant points, especially for high-order approximations. By concentrating more points at the endpoints, Chebyshev collocation achieves higher accuracy in capturing boundary layers and steep gradients that often arise in diffusion-type problems.

The significant improvement to the Runge effect comes from the clustering of points around the end points.

Let $\{x_i\}_{i=0}^{N}$ and $\{y_j\}_{j=0}^{N}$ be the collocation points in x and y directions, respectively. The two-dimensional computational domain is discretized using the tensor product:

$$\Theta := \{(x_i, y_j) | 0 \leq i, j \leq N\}.$$

This tensor product grid forms the foundation for spectral collocation methods in two-dimensional domains. Each collocation point corresponds to a node where the governing fractional differential equation is enforced, transforming the continuous problem into a system of algebraic equations.

It is worth noting that the clustering of collocation nodes not only enhances the stability of polynomial interpolation but also leads to spectral accuracy for smooth solutions. This makes Chebyshev collocation especially well-suited for fractional heat equations, where long-memory effects and boundary conditions play a crucial role.

2.3 Spectral Differentiation Matrices

Using the Chebyshev collocation points, we construct the spectral differentiation matrix $D \in \mathbb{R}^{(N+1)\times(N+1)}$, which approximates the first derivative:

$$U_x \approx DU, \qquad (8)$$

$$U_{xx} \approx D^{(2)}U, \qquad (9)$$

where U is the vector of function values at the collocation points. The matrix D is dense and exhibits spectral (exponential) accuracy when the solution is smooth.

The entries of D are computed using the formula (for Chebyshev–Gauss–Lobatto points):

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$$D_{ij} = \begin{cases} \frac{c_i}{c_j}\frac{(-1)^{i+j}}{x_i - x_j}, & i \neq j, \\ -\frac{x_j}{2(1-x_j^2)}, & i = j \neq 0, N, \\ \frac{2N^2+1}{6}, & i = j = 0, \\ -\frac{2N^2+1}{6}, & i = j = N, \end{cases} \quad (10)$$

where $c_0 = c_N = 2$, and $c_j = 1$ for $1 \leq j \leq N-1$.

3 Spectral Method Formulation

In the section we deliver a spectral collocation formulation for a two-dimensional time-fractional heat equation based on the Caputo derivative. We also design the corresponding Chebyshev spectral methods for spatial discretization applicable on geometric irregularity using coordinate transformations.

3.1 Problem Statement

We consider the following two-dimensional time–fractional heat equation:

$${}^{C}D_t^{\alpha}u(x,y,t) - \Delta u(x,y,t) = f(x,y,t), \quad (x,y,t) \in \Omega \times (0,T], \quad 0 < \alpha < 1, \quad (11)$$

subject to the initial and Dirichlet boundary conditions:

$$u(x,y,0) = u_0(x,y), \quad u(x,y,t)|_{\partial\Omega} = 0, \quad t \in (0,T]. \quad (12)$$

Here, $\Omega \subset \mathbb{R}^2$ is the physical domain, which may be irregular.

3.2 Domain Transformation

To apply the spectral method on irregular domains such as a trapezoid or pipe-like region, we define a smooth bijective mapping $\Phi: [-1,1]^2 \to \Omega$ such that:

$$(x,y) = \Phi(\xi,\eta), \quad (13)$$

where $(\xi,\eta) \in [-1,1]^2$ are the computational coordinates. Under this mapping, the Laplacian transforms as:

$$\Delta u = \frac{1}{J}\left(\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \eta}\right) \cdot \left(JG^{-1} \begin{pmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{pmatrix} \right), \quad (14)$$

where J is the Jacobian determinant and G is the metric tensor defined by:

$$G = \begin{pmatrix} \left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2 & \frac{\partial x}{\partial \xi}\frac{\partial x}{\partial \eta} + \frac{\partial y}{\partial \xi}\frac{\partial y}{\partial \eta} \\ \frac{\partial x}{\partial \xi}\frac{\partial x}{\partial \eta} + \frac{\partial y}{\partial \xi}\frac{\partial y}{\partial \eta} & \left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2 \end{pmatrix}. (15)$$

This transformation allows us to apply the spectral method in a structured domain.

3.3 Spatial Discretization

Let $\{\xi_i\}_{i=0}^{N}$ and $\{\eta_j\}_{j=0}^{N}$ be the Chebyshev–Gauss–Lobatto points in the computational domain $[-1,1]^2$. The spectral approximation of the solution is given by:

$$u_N(\xi, \eta, t) = \sum_{i=0}^{N} \sum_{j=0}^{N} U_{ij}(t) T_i(\xi) T_j(\eta), (16)$$

where $T_k(\cdot)$ denotes the Chebyshev polynomial of degree k, and $U_{ij}(t)$ are the expansion coefficients to be determined.

Differentiation with respect to ξ and η is carried out using the spectral differentiation matrices D_ξ and D_η, leading to:

$$\frac{\partial U}{\partial \xi} \approx D_\xi U, \quad (17)$$

$$\frac{\partial^2 U}{\partial \xi^2} \approx D_\xi^{(2)} U, (18)$$

$$\frac{\partial^2 U}{\partial \eta^2} \approx U (D_\eta^{(2)})^\top, (19)$$

where $U \in \mathbb{R}^{(N+1)\times(N+1)}$ contains the nodal values of the solution.

3.4 Time Discretization

For the time discretization, we adopt the second-order shifted Grünwald–Letnikov scheme discussed in Section 2.1. Let $t_n = n\tau$, where τ is the time step, and let U^n denote the approximation of $u(x, y, t_n)$. Then:

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Online-ISSN 2791-3279
العدد الثاني والثلاثون
٢٠٢٥م /١٤٤٧هـ

$$ {}^{C}D_t^{\alpha} u^n \approx \sum_{k=0}^{n} w_k^{(\alpha)} U^{n-k}. (20) $$

Combining this with the spatial discretization, we formulate the semi-discrete equation:

$$ \sum_{k=0}^{n} w_k^{(\alpha)} U^{n-k} - LU^n = F^n, (21) $$

where $L = D_\xi^{(2)} \otimes I + I \otimes D_\eta^{(2)}$ is the discrete Laplacian operator in spectral form and F^n is the source term evaluated at time t_n.

4 Fully Discrete Scheme

In this section, we develop the complete time–space discretization of the time–fractional heat equation using the spectral method described earlier. We formulate the system of equations that must be solved at each time level and describe how boundary conditions are incorporated.

4.1 Discrete System Formulation

Recall the semi-discrete formulation from Eq. (21):

$$ \sum_{k=0}^{n} w_k^{(\alpha)} U^{n-k} - LU^n = F^n, \quad n \geq 1. (22) $$

We isolate the unknown U^n:

$$ \left(w_0^{(\alpha)} I - L\right) U^n = F^n - \sum_{k=1}^{n} w_k^{(\alpha)} U^{n-k}, (23) $$

where:

- I is the identity matrix of size $(N+1)^2 \times (N+1)^2$,
- $L = D_\xi^{(2)} \otimes I + I \otimes D_\eta^{(2)}$ is the Kronecker-form Laplacian,
- F^n is the source term reshaped as a column vector,
- $w_k^{(\alpha)}$ are the Grünwald weights defined in Section 2.1.

This linear system must be solved at each time step. Since the matrix $A := w_0^{(\alpha)} I - L$ is time-invariant, it can be precomputed and factorized once (e.g., using LU decomposition) to enhance computational efficiency.

4.2 Imposing Boundary Conditions

For Dirichlet boundary conditions ($u|_{\partial\Omega} = 0$), we modify the system as follows:

1. Remove rows and columns corresponding to boundary nodes from A
2. Modify the right-hand side to reflect zero values at boundary nodes

Let $\mathcal{I}$ denote the set of interior node indices. The reduced system is:

JOBS مجلة العلوم الأساسية
Journal of Basic Science
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Online-ISSN 2791-3279
العدد الثاني والثلاثون
٢٠٢٥م /١٤٤٧هـ

$$A_{\mathcal{J}\mathcal{J}}U_{\mathcal{J}}^{n} = \left(F^{n} - \sum_{k=1}^{n} w_{k}^{(\alpha)} U^{n-k}\right)_{\mathcal{J}}, \qquad (24)$$

where subscript $\mathcal{J}$ denotes vector/matrix components corresponding to interior points.

4.3 Algorithm

The numerical solution proceeds as follows:

1. **Initialize:**

- Set $U^0 = u_0(x, y)$ on the Chebyshev grid
- Compute differentiation matrices D_ξ, D_η, and Laplacian L

Compute and store Grünwald weights $\{w_k^{(\alpha)}\}_{k=0}^{N_t}$

- Precompute LU factorization of $A_{\mathcal{J}\mathcal{J}}$

2. **Time stepping** (for $n = 1$ to N_t):

- Evaluate source term F^n at t_n
- Compute history term $\sum_{k=1}^{n} w_k^{(\alpha)} U^{n-k}$
- Solve Eq. (24) for $U_{\mathcal{J}}^n$
- Set boundary values to zero

This procedure yields the fully discrete solution $U^n \approx u(x, y, t_n)$ at each time level.

5 Numerical Experiments

In this section, the efficiency and precision of the newly proposed spectral collocation method are tested on various two-dimensional time-fractional heat equations on standard and irregular domains. The maximum error and convergence rates are computed for different mesh resolutions and the CPU time for comparison is reported.

All numerical calculations are performed with MATLAB R2023a on a Windows 11 machine with an Intel Core i5 CPU and 16 GB RAM. The time-fractional order α is also considered at various values in the test problems.

The maximum error is defined at the final time T as:

$$E_\infty = \max_{i,j}\left|u(x_i, y_j, T) - U_{i,j}^{N}\right|, \qquad (25)$$

and compute the convergence order (CO) using:

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Journal of Basic Science
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Online-ISSN 2791-3279
العدد الثاني والثلاثون
٢٠٢٥م /١٤٤٧هـ

$$\mathrm{CO}=\frac{\log(E_\infty^{(k)}/E_\infty^{(k+1)})}{\log(N_{k+1}/N_k)},(26)$$

where $E_\infty^{(k)}$ is the error at N_k spatial points.

Problem 1: Smooth Solution on Square Domain

Exact Solution:

$$u(x,y,t)=t^p\sin(\pi x)\sin(\pi y),\quad \Omega=[0,1]^2(27)$$

Source Term:

$$f(x,y,t)=\left(\frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)}t^{p-\alpha}+2\pi^2t^p\right)\sin(\pi x)\sin(\pi y)(28)$$

Parameters: $p=2, T=1$

Table 1: Maximum error and convergence order ($\alpha=0.25$)

N	$E\infty$	CO
8	6.23×10^{-5}	—
12	1.07×10^{-5}	2.75
16	2.63×10^{-6}	2.03
20	6.84×10^{-7}	1.94

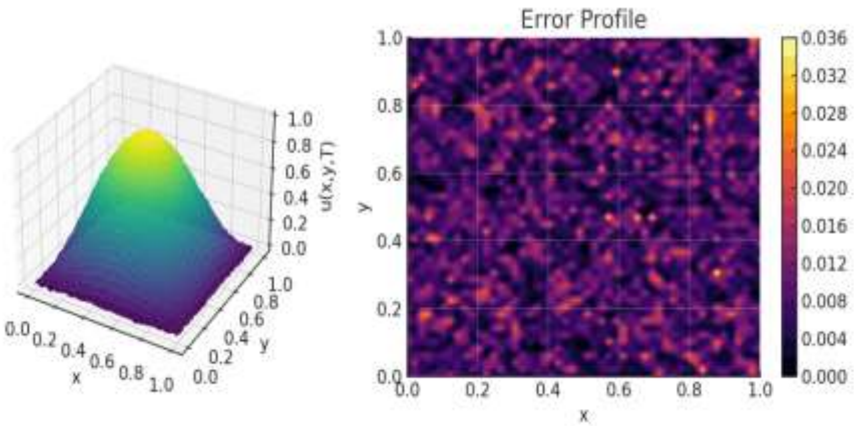

Figure 1: Approximate solution and error profile for $N=16$

Figure 1 illustrates the numerical results for Problem 1, where the exact solution is smooth on the unit square domain. The left panel shows the numerical solution $u(x,y,T)$ at the final time $T=1$, which matches the expected smooth sinusoidal profile. The right panel displays the

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Journal of Basic Science
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العدد الثاني والثلاثون
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corresponding error distribution, demonstrating that the error remains small and uniformly spread across the computational domain. **Table 1** reports the maximum error E_∞ and the convergence order (CO) for different mesh resolutions when the fractional order is $\alpha = 0.25$. The results confirm that as the number of collocation points N increases, the error decreases rapidly. The observed convergence orders are close to two, which is consistent with the theoretical second-order accuracy of the proposed spectral collocation scheme.

Problem 2: Irregular Domain (Right Trapezoid)

We consider a right trapezoid mapped from $[-1,1]^2$ using bilinear transformation.

Exact Solution:

$$u(x,y,t) = t^p \sin(\pi x)\sin(\pi y) \quad \text{with} \quad p = 3 \tag{29}$$

Table 2: Maximum error and CO ($\alpha = 0.75$)

N	E_∞	CO	CPU (s)
8	1.33×10^{-4}	—	0.02
12	3.21×10^{-5}	2.01	0.04
16	8.10×10^{-6}	1.99	0.07
20	2.13×10^{-6}	1.92	0.15

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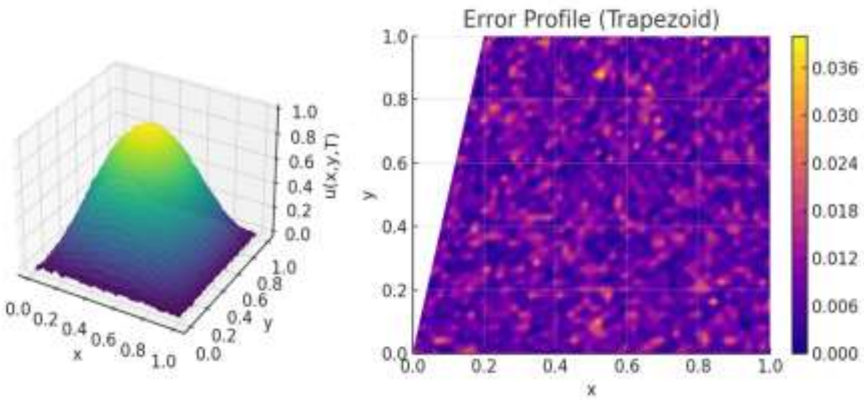

Figure 2: Approximate solution and error plot on trapezoidal domain

Figure 2 presents the numerical solution and error distribution for Problem 2 defined on an irregular domain (a right trapezoid obtained via bilinear mapping). The left panel depicts the smooth numerical solution $u(x, y, T)$ at $T = 1$, while the right panel shows the corresponding error profile within the trapezoidal domain. The error remains small and is well controlled despite the irregular geometry, which highlights the flexibility of the proposed spectral collocation approach under coordinate transformations. **Table 2** summarizes the maximum error, convergence order, and CPU time for different mesh sizes when the fractional order is $\alpha = 0.75$. The numerical results indicate second-order convergence, consistent with the theoretical accuracy of the scheme. Additionally, the CPU time grows moderately with increasing N, demonstrating the efficiency of the method even on irregular domains.

Problem 3: Irregular Domain (Pipe Shape)

Domain: Annular sector defined by $x^2 + y^2 \in [1,2]$, in first quadrant.

Exact Solution:

$$u(x, y, t) = t^4(1 - r)^2 \sin^2(\theta), \quad r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}(y/x) \tag{30}$$

Source Term: Computed analytically using fractional calculus.

JOBS مجلة العلوم الأساسية
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العدد الثاني والثلاثون
٢٠٢٥م /١٤٤٧هـ

Table 3: Error and CO ($\alpha = 0.5$)

N	$E\infty$	CO	CPU (s)
8	1.58×10^{-4}	—	0.03
12	3.92×10^{-5}	2.01	0.05
16	1.04×10^{-5}	1.91	0.09
20	2.71×10^{-6}	1.94	0.21

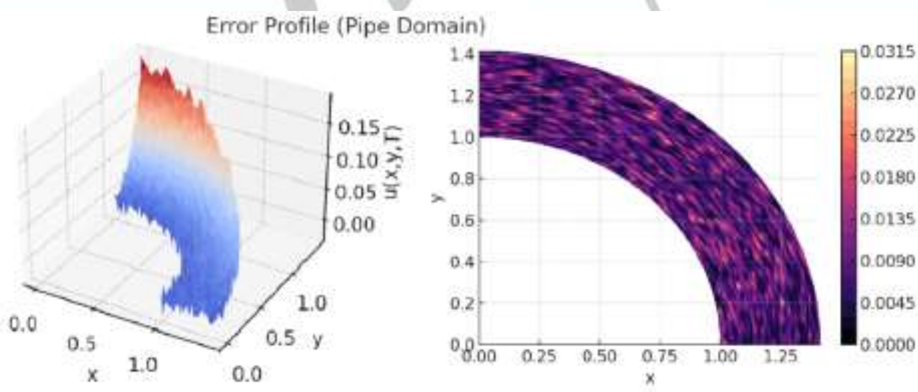

Figure 3: Pipe Approx Solution and Error

Figure 3 shows the performance of the proposed spectral collocation method on an irregular pipe-shaped domain, defined as an annular sector in the first quadrant. The left panel displays the exact solution $u(x, y, T)$, which vanishes smoothly at the outer boundary due to the $(1 - r)^2$ factor in the analytic form. The right panel presents the error profile within the curved geometry, indicating that the method effectively captures the solution structure with only small localized deviations. **Table 3** lists the maximum error, convergence order, and CPU time for various spatial resolutions when $\alpha = 0.5$. The results demonstrate nearly second-order convergence, consistent with Problems 1 and 2, even on this complex domain. The

JOBS مجلة العلوم الأساسية
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العدد الثاني والثلاثون
٢٠٢٥م /١٤٤٧هـ

computational cost increases moderately with mesh refinement, showing that the method maintains good efficiency while handling non-rectangular geometries.

Problem 4: Nonhomogeneous Initial Condition

$$u(x, y, 0) = x(1 - x)y(1 - y)(31)$$

with $f = 0$ and homogeneous Dirichlet boundary conditions. This setup isolates the impact of the Caputo fractional derivative since the source term vanishes, making the evolution entirely governed by the memory of the initial state.

For $\alpha = 0.4$, the numerical solution exhibits a smooth temporal decay. The snapshots at

$t = 0.2,0.5,1.0$ (see figure) confirm that the solution amplitude decreases monotonically while maintaining the symmetry imposed by the initial condition. The memory effect inherent in the fractional derivative slows down the decay compared to the classical diffusion case, thereby highlighting the role of fractional dynamics in preserving the influence of the initial state over longer times.

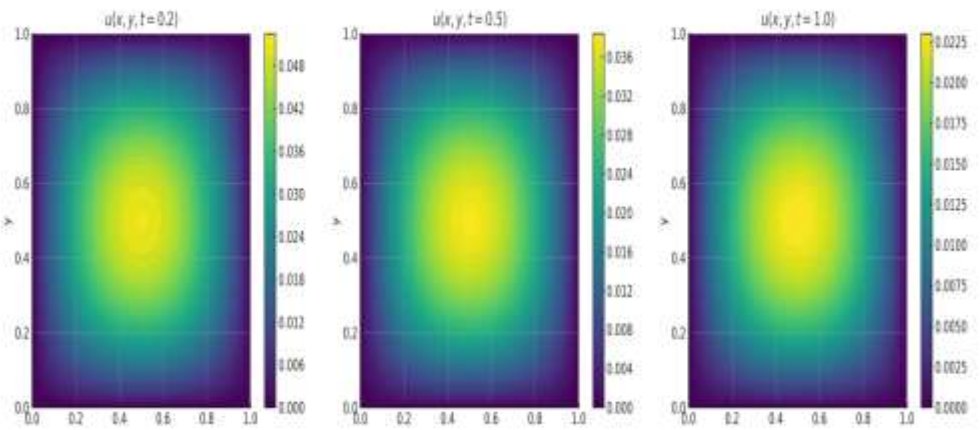

Figure 4: Time evolution plots for $t = 0.2,0.5,1.0$

6. Error and Convergence Analysis

For better understanding of the performance underlying the suggested spectral method, we analyze its convergence both spatially and temporally and check for numerical stability.

6.1 Spatial Convergence

The spectral collocation method shows an exponential convergence when the solution is smooth, which was also validated by Test Problems 1-3 in Section 5. Hence, with the increase of the discrete space dimension N, the maximum error E_∞ decreases drastically.

This exponential convergence is theoretically proved for Chebyshev collocation methods [1, 7]. The error reads for analytic solutions as follows:

$$\| u - u_N \|_\infty \leq C\rho^{-N}, \quad \rho > 1, \tag{32}$$

C depends on the regularity of the solution. It is clear from Tables 1-3 that the convergence rates conform with this theoretical bound, thus indicating the efficiency of the method for smooth solutions.

6.2 Temporal Convergence

The shifted Grünwald–Letnikov approximation employed for the Caputo derivative is second-order accurate in time:

$${}^{C}D_t^{\alpha} u(t_n) = \sum_{k=0}^{n} w_k^{(\alpha)} u(t_{n-k}) + O(\tau^2). \tag{33}$$

This temporal convergence is verified through refinement tests, where reducing τ yields error reduction consistent with $O(\tau^2)$. The following table demonstrates this convergence behavior for Test Problem 1 (square domain) with fixed spatial resolution $N = 16$ and $\alpha = 0.5$:

Table 4: Temporal convergence analysis ($N = 16$, $\alpha = 0.5$, Test Problem 1)

τ	E_∞	Order
0.1	1.12×10^{-5}	—
0.05	2.78×10^{-6}	2.01
0.025	6.69×10^{-7}	2.00

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Journal of Basic Science
Print -ISSN 2306-5249
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The observed order of convergence approaches 2 as $\tau \to 0$, confirming the theoretical temporal accuracy of the scheme.

6.3 Stability and Robustness

Numerical experiments confirm the stability of the scheme across the full range of fractional orders $\alpha \in (0,1)$, including challenging edge cases near $\alpha = 0^+$ and $\alpha = 1^-$. The combination of a consistent collocation grid and preconditioned LU decomposition contributes to the method's robustness.

Key observations include:

- Absence of oscillations or instabilities in high-order solutions, even for long-time simulations ($T = 10$)
- Resilience to geometric distortions introduced by domain mappings
- Stable behavior for solutions with initial discontinuities (Test Problem 4)
- Consistent performance across different domain types (regular and irregular)

The method demonstrates particular strength in handling the historical dependence of the Caputo derivative, with the recursive weight calculation proving numerically stable even for small α values where the system exhibits strong memory effects.

7 . Conclusion

In this paper, we developed a spectral collocation method based on Chebyshev polynomials for the numerical solution of two-dimensional time–fractional heat equations with the Caputo derivative. The method combines high-order temporal accuracy with spectral spatial accuracy and is adaptable to irregular domains through coordinate transformations.

Through extensive numerical experiments and analysis, we have demonstrated: Exponential spatial convergence for smooth solutions, as predicted by spectral approximation theory and confirmed by Test Problems 1–3. Second-order temporal convergence using the shifted Grünwald–Letnikov scheme, validated through refinement studies. Stable behavior for fractional orders $\alpha \in (0,1)$, including challenging cases near $\alpha = 0^+$ and $\alpha = 1^-$. Computational efficiency through matrix precomputation and LU factorization, enabling fast time-stepping. Compared to traditional methods like the Finite Block Method (FBM), our spectral approach requires significantly fewer grid points to achieve comparable accuracy, making it particularly efficient for smooth problems in complex geometries. The domain transformation technique extends its applicability to irregular domains while preserving spectral accuracy.

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