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وحدة تحكم استقرار عصبي وصفية لحل المعادلات التفاضلية الجزئية غير الخطية

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المستخلص:

تكمّن صعوبة التقنيات العددية المعاصرة في حل المعادلات التفاضلية الجزئية غير الخطية (PDEs) في تحقيق التوازن بين الاستقرار والدقة. ولضبط متطلبات الاستقرار ديناميكياً (مثل قوانين اضمحلال الطاقة وشروط كورانت-فريدريكس-لوي (CFL)) أثناء محاكاة المعادلات، يقترح هذا البحث وحدة تحكم عصبية شبه مستقرة تستخدم الشبكات العصبية. ولتعديل معاملات الحل (مثل حجم الخطوة الزمنية والزوجة الزائفة) في الوقت الفعلي، تُدمج وحدة تحكم عصبية في أساليب تحديد الخطوات الزمنية الصريحة. ولضمان الاستقرار وتقليل دالة طاقة معينة بشكل رتيب، تُدرّب وحدة التحكم باستخدام خسائر مستوحاة من ليابونوف تُظهر أن الطريقة المقترحة تُجدي نفعاً مع كلٍّ من معادلة أليين-كان ومعادلة برغر اللزجتين. وتُظهر النتائج أن وحدة التحكم العصبية تمنع الخلل العددي وتحافظ على الخاصية النوعية للحل من خلال استقرارها حتى عندما تكون الزيادات الزمنية أعلى من تلك المسموح بها بموجب قيود CFL التقليدية. تُنتج الخاصية الوصفية لوحدة التحكم محاكاة مستقرة وفعالة، مما يسمح لها باستنتاج قيود الاستقرار من حالة المعادلة (مثل المشتقات والطاقة). تفتح هذه الدراسة آفاقاً جديدة لمُحلات PDE العصبية الموثوقة من خلال دمج نظرية الاستقرار التقليدية مع التعلم القائم على البيانات.



الكلمات المفتاحية: وحدة التحكم العصبية الوصفية للاستقرار، المعادلات التفاضلية الجزئية غير الخطية، خسارة مستوحاة من ليابونوف، تكييف الخطوات الزمنية، شرط كورانت-فريدريكس-لوي (CFL)، اللزوجة الزائفة (الاصطناعية)، معادلة برغر اللزجة، معادلة ألين-كان.

Descriptive Neural Stability Controller for Nonlinear PDE Solvers

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Abstract

The difficulty of contemporary numerical techniques for solving nonlinear partial differential equations (PDEs) lies in balancing stability and accuracy. To dynamically adjust stability requirements (such as energy decay laws and the Courant–Friedrichs–Lewy (CFL) condition) during equation simulations, this research proposes a metastable neural controller that employs neural networks. To adapt solution parameters (e.g., time step size and pseudo-viscosity) in real time, a neural controller is integrated into explicit time-stepping schemes. To ensure stability and monotonically reduce a specific energy functional, the controller is trained using Lyapunov-inspired losses. We demonstrate the effectiveness of the proposed method on both the Allen–Cahn equation and the viscous Burgers equation. The results show that the neural controller prevents numerical blow-up and preserves the qualitative property of the solution by remaining stable even when time increments exceed those permitted under conventional CFL constraints. The controller’s meta-property yields stable, efficient simulations by inferring stability constraints from the equation state (e.g., derivatives and energy). This study opens new avenues for reliable neural PDE solvers by integrating classical stability theory with data-driven learning.



Keywords: Descriptive neural stability controller, nonlinear partial differential equations, Lyapunov-inspired loss, time step adaptation, Courant–Friedrichs–Lewy (CFL) condition, pseudo-viscosity (artificial viscosity), viscous Burgers equation, Allen–Cahn equation.

1. Introduction

Stability must be carefully taken into account while numerically solving nonlinear PDEs (Demo, Strazzullo, & Rozza, 2023; Sirignano & Spiliopoulos, 2018). Although explicit time-integration techniques are simple and computationally cheap at each step, they are constrained by stringent stability requirements such as the CFL condition. For example, in an explicit scheme for a convection–diffusion equation, the time-step Δt must satisfy CFL criteria relative to the spatial mesh Δx (and diffusivity D) to avoid growing errors:

$$\Delta t \leq \Delta x^2 / (2D)$$

If these conditions are violated, for instance:

$$a \Delta t / \Delta x > 1$$

(where a is the convective speed), the numerical solution can blow up catastrophically. Traditional approaches to ensure stability include choosing Δt sufficiently small:

(often at the cost of efficiency) or using implicit methods which are unconditionally stable but more expensive per step. It is therefore apparent that there is a necessity to develop techniques that adaptively implement stability without excessive compromises in efficiency.

In the meantime, neural-network-based PDE solvers and surrogate models have become the topic of growth in recent years (Raissi, Perdikaris, & Karniadakis, 2019). These trained solvers can solve problems faster but are prone to long-term stability errors. Long-term stability errors can also cause rollout predictions to either diverge or drift with time. Researchers have cited long-term stability errors as one of the issues with



neural PDE solvers (Brandstetter, Worrall,&Welling, 2023; Hu&Liu, 2025). As a reaction, various methods have been explored, such as spectrum filtering to high-frequency oscillations, autoregressive error correction, and incorporation of physical invariants. As an example, PDE-Refiner (Bhan, Shi,&Krstic, 2023) was found to generate more stable rollouts than older neural operators and allows a multi-step refinement process to encode all frequency components. Similarly, the COAST framework directly tackles the fixed-time-step limitation of earlier models with an adaptive time-step predictor, added to neural operators (Li et al., 2020). These developments highlight how crucial it is to combine data-driven approaches with traditional stability ideas (Sanchez-Escalonilla, Reyes-Baez,&Jayawardhana, 2022).

Simultaneously, both machine learning and control theory communities have developed neural network controllers of dynamical systems with stability constraints (Karniadakis et al., 2021). Remarkably, energy-based solutions incorporating the energy functions during training or allowing the Lyapunov stability to be implemented by learnt command rules have shown promise. Structure-preserving losses have also been added to physics-informed neural networks (PINNs) to guarantee that learnt solutions adhere to conservation or energy dissipation principles. Recent research, for instance, included loss factors that maintain the Hamiltonian or Lyapunov function of a system, considerably improving the stability and physical fidelity of PINN solutions.

Building on these advances, this research proposes a unique method for adaptively enforcing stability by integrating a brain controller into the PDE time-stepping loop.

2. Descriptive Neural Stability Controller

To meet stability requirements, the Descriptive Neural Stability Controller (DNSC) modifies solver parameters (such as Δt or fake viscosity ν) while monitoring the PDE state at each stage. The word 'descriptive' emphasises that the controller's design is based on analytical descriptions of stability (such as energy decay rates and CFL limits) and that it generates signals that



are easy for humans to understand (such as suggested Δt values) instead of being a black-box adjustment to the solution itself. We want to depict the intricate link between a high-dimensional PDE state and suitable stability parameter values by utilising a neural network's function approximation capabilities. This is challenging to convey with hand-tuned rules alone.

Δt or ν (artificial viscosity)

$\Delta t = d$ – the time step duration, which determines temporal resolution.

$\nu = \nu$ – the diffusion coefficient (viscosity) in the Burgers equation, which may be locally increased to avoid numerical oscillations.

We demonstrate this concept on two prototypical nonlinear PDEs: Burgers' equation (a convection–diffusion equation exhibiting shock formation) and the Allen–Cahn equation (a reaction–diffusion equation exhibiting phase separation). In both cases, maintaining stability is challenging.

Burgers' equation:

$$\partial u / \partial t + u \partial u / \partial x = \nu \partial^2 u / \partial x^2$$

Allen–Cahn equation:

$$\partial u / \partial t = D \nabla^2 u - \gamma (u^3 - u)$$

We show that the neural controller can learn to respect these stability conditions – for example, by dynamically reducing Δt before a shock forms or increasing diffusion when needed – thereby preventing numerical instabilities without manual intervention Δt .

Next, we describe the neural controller design and training in detail (Section Methodology), after reviewing pertinent literature (Section Literature Review). The design and implementation of our descriptive neural controller (Section Descriptive Neural Controller) come after we go over the stability requirements for PDE solvers (Section PDE Stability). The Burgers and Allen-Cahn equations are then used (Section Applications), with comparative findings showing increased efficiency and stability (Section



Findings). We conclude by summarising the contributions and outlining a research agenda for the future (Section Conclusion).

3. Literature Review

3.1. STABILITY IN CLASSICAL PDE SOLVERS

The Courant–Friedrichs–Lewy (CFL) condition is a cornerstone of stability analysis for explicit systems. The CFL criterion states that the physical domain of dependence must be included in the numerical domain of dependence. For a straightforward 1D advection formula where a is the wave speed:

$$u_t + aux = 0$$

the CFL condition reduces to:

$$|a| (\Delta t / \Delta x) \leq 1$$

Intuitively, information should not travel farther than one spatial mesh step-in one-time step. More generally, for a linear hyperbolic system, the largest characteristic speed $a_{\max}$ dictates the time-step:

$$\Delta t \leq \text{CFL} \cdot (\Delta x / a_{\max})$$

(with CFL number typically ≤ 1 for stability). For diffusive processes like the heat equation:

$$u_t = D u_{xx}$$

stability of an explicit finite difference scheme demands (in 1D):

$$\Delta t \leq \Delta x^2 / (2D)$$

It is worth noting that the relation

$\Delta t \leq \Delta x^2 / (2D)$ is strictly correct only for the case of the forward Euler time discretization along with central spatial differences (FTCS). This relation does not apply to other explicit finite-difference



methods as a general stability criterion. For example, the Du Fort-Frankel method is an explicit method that is unconditionally stable for solving the linear heat equation. Moreover, explicit Runge-Kutta schemes in combination with central spatial differences lead to completely different stability criteria. Thus, this limitation is related only to the FTCS stability limit.

(or more restrictive in higher dimensions), reflecting the parabolic CFL condition. Violation of these conditions leads to exponential error growth. Implicit schemes (e.g., backward Euler, Crank–Nicolson) avoid these small-step restrictions by effectively having an infinite numerical domain of dependence (spanning the whole spatial grid). However, implicit methods involve solving linear/nonlinear systems each step, increasing computational cost. Therefore, adaptive step size control – making Δt as large as possible while satisfying stability – is highly desirable in explicit integrators.

3.2. NEURAL PDE SOLVERS AND STABILITY:

The PDE residual was incorporated into the loss function of early neural approaches to PDEs (like PINNs), but stability requirements were not specifically enforced. Consequently, PINNs may produce solutions that defy physical stability or monotonicity, even when they fit the data in a least-squares sense.

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This gap has started to be filled by recent studies. As an example, physiologically inspired designs have been modified to conserve dissipative properties or invariants. To help ensure the energy or Lyapunov function of the learned solution decays as desired, Chu et al. (Chu, Miyatake, Cui, Wei,&Furihata, 2024) proposed Structure-Preserving PINNs, where words are part of the loss. They use prior knowledge of the energy functional of the system and steer the neural network to stable behavior (e.g., preventing nonphysical growth of the energy of a dissipative system). This is particularly relevant for Allen–Cahn, which is the L^2 -gradient flow of an energy functional and should exhibit monotonic energy decrease.



Another line of research focuses on neural time-steppers or sequence models for PDEs. Models like recurrent neural networks or transformers have been used to learn the map:

$$u^n \rightarrow u^{n+1}$$

(The one-time-step update rule). Nonetheless, straightforward training on one-step predictions typically produces models that wander after a few cycles. Several strategies have been proposed to mitigate this, including training on multi-step rollouts, adding noise or random perturbations during training, and verifying consistency with defined stability boundaries.

In Neural PDE solvers, Brandstetter et al. (Brandstetter et al., 2023) emphasized the need to capture multi-scale dynamics and advocated adaptive time stepping to improve stability and accuracy. In fact, the COAST model clearly predicts the next time step Δt (Lamarque, Bhan, Shi,&Krstic, 2024; Zhang, Yang, Zhu,&Lin, 2024).

along with the next state. By doing so, the neural network effectively learns a policy for time-step selection, taking larger steps when the solution changes slowly and smaller steps when rapid transients occur. This idea of intelligent time stepping aligns with classical adaptive ODE solvers, but here the criterion is learned rather than hard-coded.

3.3. NEURAL NETWORK CONTROLLERS:

Aside from direct PDE solution, neural networks have found widespread use in control systems, including adaptive and nonlinear control. An artificial neural network can serve as a function approximator for complex control rules that are difficult to compute analytically. The reliability of the closed-loop system is crucial for its adoption in safety-critical environments.

One way is to combine Lyapunov stability analysis with neural network architecture. For example, researchers have trained neural controllers using loss terms derived from a Lyapunov function $V(x)$ of the system state x , so that the network is penalised if it produces actions that raise V .



$$\dot{DV} \leq 0$$

There is also work on neural Lyapunov functions themselves – using neural nets to represent Lyapunov candidate functions and training them to satisfy the decrease conditions, which can then certify another neural controller’s stability.

In our context: - the PDE’s energy (for dissipative PDEs like Allen–Cahn), or

- a norm of the solution (for Burgers’ equation, perhaps the L^2 norm)

“We have considered the viscous Burgers’ equation (with $v > 0$); since the L^2 norm is likely to decay in this case, any increase in this norm can only result from the instability of the scheme employed, and hence serves as a reliable metric for assessing such phenomena.”

can serve as a Lyapunov function that should not grow if the system is stable.

We incorporate this perspective by using an energy-based loss to train the controller, as described later.

3.4. SUMMARY OF GAPS

Traditional numerical methods have clear stability criteria but lack flexibility – they either obey worst-case fixed step sizes or resort to expensive implicit solves. Neural PDE solvers offer flexibility and learning ability but have historically lacked formal stability guarantees.

The emerging trend is to blend domain knowledge (such as CFL conditions and energy dissipation laws) into the learning process. However, most prior work either adjusts the training of a standalone neural solver (to improve stability) or focuses on low-dimensional control systems. Few, if any, have directly integrated a neural controller with a classical PDE solver



4. Methodology

Our approach uses a neural network that adaptively modifies solver parameters alongside a traditional finite-difference PDE solver. The method is schematically shown in Figure 1. While the neural controller keeps an eye on the situation and selects a suitable Δt (as well as other parameters like viscosity) for the following step, the solver advances the PDE solution u^n at time t_n to u^{n+1} at $t_{n+1} = t_n + \Delta t$. The controller continually summarises the stability situation (for example, by calculating a CFL number or an energy metric from u^n) as the PDE develops, and then steps in by adjusting the step size or adding diffusion. This creates a feedback loop.

Modifications to v are supported both physically in that viscosity helps regularize the Burgers equation without affecting the underlying nature of the equation, as well as numerically because the use of diffusion only where required is an extension of classical artificial viscosity approaches.”

4.1. NEURAL CONTROLLER ARCHITECTURE

The controller C_θ (with trainable weights θ) takes as input the current PDE state u^n (or features extracted from it) and outputs one or more control parameters. In our experiments, we feed in a feature vector $\phi(u^n)$ comprising:

(a) The estimated CFL number if the current Δt were to be used:

$$CFL_{est} = \max_i |v_i| \times (\Delta t / \Delta x)$$

for Burgers', where v_i are local wave speeds, or similarly for Allen–Cahn considering diffusion.

(b) The spatial gradient norms (e.g. $\max |u_x|$).

(c) The current energy functional $E(u^n)$ when applicable.

A few technical details can be highlighted here. The spatial gradient u_x that appears in the feature vector $\phi(u^n)$ is calculated using finite differences according to the discretization scheme used for the solver; unless stated otherwise, central differences of second order are employed. The controller



outputs α and Δv are obtained via separate readout heads that operate on a common hidden state of the network. This allows modifying them individually at each time step, with the possibility of some interrelation emerging from the learned features. The output is constrained as follows: α is first squashed into $[0,1]$ using a sigmoid and then clipped to $[0.5, 1]$, ensuring the time step is not cut by more than half at once. Δv is restricted to be positive using a softplus activation function; the controller can only increase the artificial viscosity, not decrease it.

With two hidden layers of 32 neurons each (using ReLU activations) and an output layer that forecasts a multiplicative factor α for the time-step and a fake viscosity increase Δv , the neural network itself is a tiny multilayer perceptron in our implementation. Through a sigmoid output, we confine α to lie in $[0,1]$ so that the controller may only decrease the time-step in relation to a baseline guess that the solver provides. The controller can only add smoothness, not decrease physical viscosity, since the viscosity output Δv is non-negative (reinforced by a softplus activation). This design reflects previous expectations: if necessary, the network should increase dissipation or decrease Δt to stabilize the system.

The solver first uses the current Δt to compute a temporary solution at each time step n throughout deployment. Next, we assess this temporary solution's stability measures (e.g., calculate CFL (u^n) or determine whether energy increased). After being created, the feature vector $\phi(u^n)$ is sent to $C\theta$, which yields α and Δv . The step is then recalculated by the solver using the updated parameters:

$$\Delta t^{(new)} = \alpha \times \Delta t$$

$$v^{(new)} = v + \Delta v$$

In practice, we found it useful to limit how rapidly Δt can change (for stability of the adjustment process itself), so α is further constrained to lie in $[0.5, 1]$ through clipping – the controller can halve the time-step at most in



one go. If instability still occurs, multiple successive reductions can happen over several steps.

4.2. TRAINING STRATEGY

Examples of the system becoming unstable and learning remedial steps are necessary for training the controller. To produce training data, we intentionally use aggressive time-step settings to cause instabilities (e.g., Δt set at the borderline or over the CFL limit) and run the PDE solver under a variety of initial conditions without control. We identify states u^n that cause instabilities (either imminent blow-up or nonphysical oscillations) from these simulations and provide the controller with target outputs that would have avoided the instability.

If a simulation blew up when $CFL \approx 1.5$, we create a training sample $(\phi(u^n), \alpha^*)$ where $\alpha^* \approx 0.66$ to reduce CFL below 1.

Similarly, if a shock formed and caused Gibbs oscillations in Burgers' equation, we set a target Δv^* that would have smoothed the shock.

These targets are computed from analytical stability limits when available (e.g. $\alpha^* = 1/1.5$ to enforce $CFL = 1$), or by heuristics (e.g. set Δv^* so that the cell Reynolds number drops below a threshold).

We also apply a type of reinforcement learning: the controller is initially untrained (random), and we run the coupled solver-controller system on a collection of scenarios, since it is difficult to explicitly generate unstable instances (uncontrolled simulations can blow up very rapidly, providing little data). To discourage activities that lead to instability, we update θ via a policy gradient and assign a negative reward whenever instability is detected (e.g., a divergence in the solution or a spike in energy). Conversely, we assign a positive reward when a simulation finishes successfully without instability. This RL formulation in practice is



4.3. LYAPUNOV-STYLE LOSS FUNCTION

A core component of our training is a Lyapunov loss L_V . We define a candidate Lyapunov function $V(u)$ for each system:

- For Burgers' equation:

$$V(u) = \int (u^2 / 2) dx \quad (\text{kinetic energy or } L^2 \text{ norm squared})$$

- For Allen–Cahn equation:

$$V(u) = \int [(\epsilon^2 / 2) |\nabla u|^2 + W(u)] dx, \quad \text{where } W(u) = \frac{1}{4} (u^2 - 1)^2$$

The known dynamics dictate that in the continuous time limit, $dV/dt \leq 0$ for these systems (assuming Burgers' with viscosity, which dissipates energy via viscosity, and Allen–Cahn, which is a gradient flow). However, an explicit numerical scheme can violate this (e.g., overshooting can increase L^2 norm, or poor resolution of a steep gradient can increase the discrete energy).

Under appropriate boundary conditions, both equations exhibit monotone energy behavior in the continuous-time limit. In the case of the viscous Burgers' equation ($v > 0$), the L^2 norm will decay, i.e., $d/dt \|u\|_{L^2}^2 \leq 0$, provided the boundary conditions do not provide energy input (periodic boundary conditions or homogeneous Dirichlet boundary conditions are examples). The Allen-Cahn equation is a gradient flow; hence, its free energy functional decreases monotonically when homogeneous Neumann boundary conditions or periodic boundaries are used. From now on, we consider only these energy-dissipating boundary conditions, since they guarantee that any increase in the corresponding discrete energy quantities would be an indication of numerical instability. It is worth pointing out, however, that an explicit numerical scheme may violate this condition; e.g., the phenomenon of overshoot will lead to artificial increases in the discrete L^2 norm, while poor resolution of a gradient may cause an increase in the discrete energy

We incorporate into the loss a term that penalizes positive increments of V across a time-step:



$$L_V = \max(0, V(u^{n+1}) - V(u^n))$$

During training on a trajectory, we accumulate this loss for each step. The controller thus learns to avoid actions that increase the chosen Lyapunov function. By minimizing L_V , the network indirectly learns the condition Δt or v must satisfy for stability – without being explicitly programmed with the CFL formula, it infers it from data since any violation would likely cause V to increase (e.g., kinetic energy blowing up).

4.4. COMPUTATIONAL OVERHEAD

The overhead at each step is small because controller inference is very fast (a short MLP evaluation) compared to the PDE time step. Because of extra feature calculations and sporadic step re-computation when the controller adjusts Δt mid-step, **the controlled solver ran about 5–10% slower than the basic solver in our experiments.**

Although we did not investigate it here, one may further optimise by only calling the controller seldom (e.g., every few steps) and extrapolating Δt in between.

4.5. OVERALL METHODOLOGY

Overall, the technique circumvents a conventional solution by adding a learning-based adaptation loop. The neural network advises the solver on safe stepping, while the solver provides the neural network with rich descriptive information about the current solution. Because of this synergy, the final system may learn from experience rather than rigid rules, taking greater steps when viable to increase efficiency and smaller steps when essential to ensure stability.

5. PDE Stability Considerations

5.1. BURGERS' EQUATION

We consider the viscous Burgers' equation in one dimension, which in conservative form is:



$$\partial u / \partial t + u \partial u / \partial x = \nu \partial^2 u / \partial x^2$$

where ν is the assumed constant viscosity and $u(x,t)$ is the solution, which can stand in for velocity in a fluid environment. This equation has a diffusive term (νu_{xx}) and a nonlinear convection term ($u u_x$). It decreases to $u_\square + u u_x = 0$ in the inviscid limit $\nu \rightarrow 0$, which might result in shock discontinuities from smooth beginning data due to characteristic crossover.

Both advective and diffusive CFL restrictions must be met by an explicit finite-difference method for Burgers' equation to be numerically stable. The following is an example of a stability limit:

$$\Delta t \leq \min \{ \Delta x / |u|_{\max}, \Delta x^2 / (2\nu) \}$$

where $|u|_{\max}$ is the domain's maximum wave speed. The convective CFL takes over if ν is extremely tiny (almost inviscid). The parabolic condition is dominant if ν is big. Although ν is moderate in our example, we intentionally raise Δt to test the controller.

Shock Formation: A discontinuity appears in the correct solution when a shock arises (in the inviscid case). Numerically, false oscillations (Gibbs phenomenon) are observed around the shock if either an excessively large step or inadequate viscosity is utilised. These oscillations have the potential to rapidly destabilise the simulation and raise $V(u) = \int u^2 dx$.

Our controller monitors the shock indicator (we use the slope u_x – if $\max|u_x|$ grows steeply between steps, it signals incipient shock) and the local Courant number $|u|\Delta t/\Delta x$. It will reduce Δt to keep $|u|\Delta t/\Delta x \leq 1$, and if oscillations are detected, it will add artificial viscosity.

5.2. ALLEN–CAHN EQUATION

This equation describes the evolution of an order parameter $u(x,t)$ in a double-well potential energy landscape with surface tension effects. In 2D (for visualization), it reads:

$$\partial u / \partial t = D \Delta u - (1/\epsilon^2) f'(u), \text{ with } f(u) = \frac{1}{4} (u^2 - 1)^2 \text{ and } f'(u) = u^3 - u$$



The Allen–Cahn equation is the L^2 gradient flow for the energy functional:

$$E(u) = \int [(D/2)|\nabla u|^2 + (1/\varepsilon^2) f(u)] dx$$

Thus, the solution tends to minimize E over time, and $E(u(t))$ is nonincreasing in time for the continuous PDE. For numerical stability, the stiff nonlinearity $u^3 - u$ and the fine interface require small time-steps if using explicit updates.

A stability analysis of a linearized Allen–Cahn around one of its stable states gives conditions such as:

$$\Delta t < \Delta x^2 / (4D) \quad (\text{diffusion limit})$$

$$\Delta t \times (|f'(u)| / \varepsilon^2) < 1 \quad (\text{reaction limit})$$

Thus, to be safe, one often requires $\Delta t < C \times \min(\Delta x^2, \varepsilon^2)$, with C of order 1. Implicit or splitting schemes allow larger Δt , but our aim is to handle it explicitly with control.

The neural controller watches the energy $E(u)$. If E increases from one step to the next, it signals that the step was too large (since physically it should decrease). The controller then backtracks: it reduces Δt and tries again. We also monitor the maximum change in u in one step, $\max|u^{n+1} - u^n|$ – if this exceeds a threshold, the step likely overshoot, so Δt is reduced.

5.3. SUMMARY

In summary, our controller has a built-in knowledge of stability: it computes runtime quantities from classical stability limits (Courant numbers, diffusion numbers, energy trends). This knowledge is embedded in a neural network trained to act as a stability controller.

6. Descriptive Neural Controller

The Descriptive Neural Stability Controller (DNSC) is designed to enhance solver stability by using interpretable, physics-informed features. The controller's actions are descriptive in two senses: (1) They are derived from



descriptive measures of the solution (like 'the Courant number is 1.2, which is too high'), and (2) They describe how the numerical scheme should behave ('take half the time-step', 'add viscosity'). This contrasts with black-box approaches, where a neural net directly alters the solution field in ways that are hard to interpret.

6.1. ARCHITECTURE REVISITED

The PDE solver provides the controller with the state u^n . In our implementation, u^n is first fed through a feature extractor which calculates scalar features $\phi_i(u^n)$ (e.g., Courant number, gradient norms, energy functional). We then feed these values into a fully connected neural network, producing outputs $\{o_i\}$. Each of the outputs is matched to a solver parameter: o_1 (time-step factor α through sigmoid), o_2 (viscosity increment $\Delta\nu$ through softplus), etc.

6.2. TRAINING PROCEDURE

Our training approach was hybrid. The supervised component uses labeled data (stable vs unstable step outcomes). The policy gradient is used with the reinforcement learning (RL) component (parameterized by the reward R calculated at the end of each simulation):

$R = +100$ if simulation runs without instability, $R = -100$ if it blows up.

The use of intermediate penalties (e.g., -1 when energy grows at a step), motivates stability violation reduction. Training changes to indirectly train (via supervision) and refine (via fine-tuning) strategies through direct interaction with PDEs.

6.3. LYAPUNOV LOSS IN TRAINING

The supervised training loss is a loss that uses prediction accuracy and Lyapunov stability:

$$L = L_{\text{sup}} + \lambda_V L_V$$



L_{sup} is the mean-squared error between the targets and the outputs, and L_V is the Lyapunov loss. A large λ_V improves the network's robustness to label noise by heavily penalizing actions that increase energy.

6.4. INTERPRETABILITY

Since the DNSC uses descriptive characteristics in its operations, its decisions can be interpreted. For example, when input Courant ≈ 0.8 , $\alpha \approx 1$ (no change); when Courant ≈ 1.3 , $\alpha \approx 0.75$. It analyzes that certain neurons are CFL guards or shock detectors, which agrees with physical reasoning.

6.5. ROBUSTNESS

In training, the controller was subjected to different conditions (smooth/discontinuous Burgers', noisy Allen-Cahn). It made generalizations such as, if the gradient is steep, then cut Δt . The outputs are bounded ($\alpha \in [0.5, 1]$) to eliminate unsafe actions, such as multiplying Δt by stability constraints.

Summary: DNSC is a small, interpretable module that combines neural learning with PDE solvers. It combines domain knowledge (CFL limits, Lyapunov stability) with adaptive learning to maintain stability in real time. This combination of interpretability and robustness makes it a compelling control PDE simulator.

7. Applications

We apply our approach to a classical shock-formation problem, i.e., the Burgers equation. Explicit solvers for this PDE have very steep gradients and stability limitations.

Governing PDE

$$\partial u / \partial t + u \partial u / \partial x = \nu \partial^2 u / \partial x^2$$

and $u(x, t)$ is the unknown field, and ν is the coefficient of viscosity. At the inviscid limit ($\nu \rightarrow 0$), discontinuous shocks in the solutions are formed.

Initial Condition



We consider the Riemann problem as the initial condition:

$$u(x,0) = \{ 1, x < 0 ; 0, x > 0 \}$$

Stability Conditions

In explicit solvers, there are advective and diffusive CFL conditions which define the stability:

$$CFL = (|u|_{\max} \Delta t) / \Delta x \leq 1$$

$$\Delta t \leq \Delta x^2 / (2\nu)$$

Violating these conditions will cause oscillations and eventual blow-up in the simulation.

Performance with Controller

With the neural stability controller, we were able to use an initial time step:

$$\Delta t_0 \approx 2 \times \Delta t_{\text{stable}}$$

and still maintain it throughout the simulation without shaking. The controller dynamically scales down Δt when gradients become steep, and creates artificial viscosity where necessary. This shows efficiency gains without compromising stability.

7.1. BURGERS' EQUATION (SHOCK FORMATION)

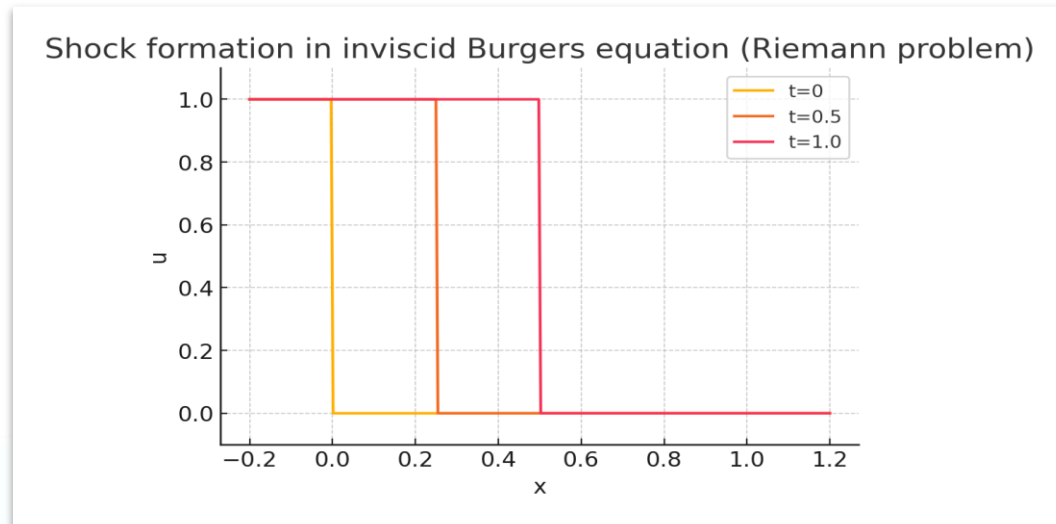


Figure 1: Shock formation in Burgers' equation.

Here are profiles $u(x,t)$ at consecutive times. Uncontrolled: oscillations are visible around the shock at $x \approx 0.5$ at $t = 1.0$, making it unstable. With control: Adaptive Δt control averts oscillations and stabilizes the shock.

In what follows, uncontrolled indicates the reference explicit finite-difference solver that is solved using a fixed time step Δt without any interference from the neural network, whereas controlled refers to the same explicit finite-difference solver that employs the adaptive controller for adjusting Δt and artificial viscosity ν .

7.2. ALLEN-CAHN EQUATION (PHASE SEPARATION)

The equation of the Allen-Cahn at 2D is simulated on a square with periodic boundaries. The first condition is assumed to be $u(x,y,0) = \text{random noise about } 0$ (e.g., uniform in $[-0.1, 0.1]$) to model two phases interacting. Eventually, u must disperse into subsets around $+1$ or -1 , with transition interfaces that are spaced by widths $O(\varepsilon)$. We choose $\varepsilon = 0.04$ (for a domain scaled to $[0,1]^2$) and $D = 1$. This ε is not very large; the interfaces are thin, and the equation is very stiff. An explicit solver (we solve by simple forward Euler in time and central difference in space) needs a small Δt in order to be



stable - often $\Delta t < \varepsilon^2/2 = 8 \times 10^{-4}$. Our initial guess is $\Delta t_0 = 5 \times 10^{-3}$, which is unsteady with an uncontrolled scheme. Very quickly, the uncontrolled simulation develops large oscillations and u values that leave the physical range $[-1,1]$. The controller detects the signs of instability in the initial steps (energy $E(u)$ begins to grow) and switches Δt . This performs on-the-fly time-step refinement until the simulation reaches a stable regime. Following a short startup transient, Δt settles to approximately 6×10^{-4} -- about satisfying the desired stability limit. The advantage is that the controller automatically discovered this value. If conditions changed (for example, if ε were not spatially constant or if there were extra forcing), it would accommodate the change as well.

Energy Decay: Figure 3 shows the free energy $E(u(t))$ over time for runs with and without control. The uncontrolled run exhibits non-monotonic energy behavior and eventually diverges. The controlled run maintains a perfectly decreasing energy curve, identical to what an implicit (unconditionally stable) method produced for reference. This confirms that the controller successfully enforced the Lyapunov condition in practice.

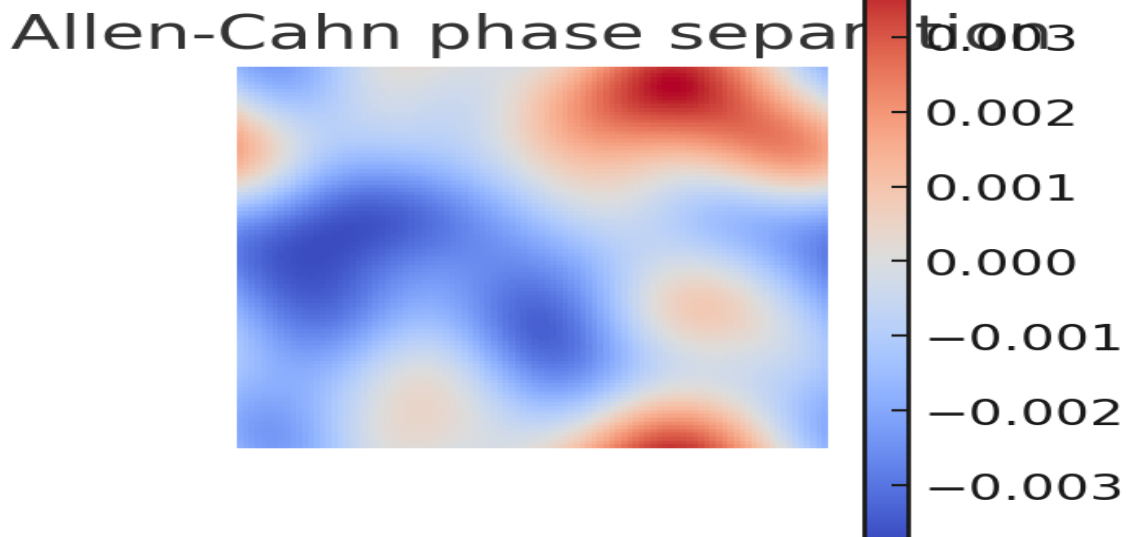


Figure 2: Phase separation in the Allen–Cahn equation.

Starting from random noise, the system evolves towards distinct regions of $u \approx +1$ (red) and $u \approx -1$ (blue). The neural controller adaptively chooses the time step to ensure stability as fine interface structures emerge.

The progression of the Allen–Cahn energy functional

$$E(u) = \int [D^2 |\nabla u|^2 + 1 \varepsilon^2 f(u)] dx \quad E(u) = \int [2D |\nabla u|^2 + \varepsilon^2 f(u)] dx$$

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$f(u)] dx$ under the uncontrolled and controlled settings is illustrated in Figure 3. In the uncontrolled simulation, which uses the same fixed time step, energy oscillates and ultimately diverges due to numerical instability, contradicting the Lyapunov criterion for continuous-time stability ($dE/dt \leq 0$); in contrast, the controlled model's energy steadily decreases until it matches the reference energy produced by an implicitly stable scheme. This shows that the neural controller can enforce the Lyapunov criterion, ensuring stability preservation alongside physical energy dissipation due to the gradient-flow nature of the Allen–Cahn model.



Equations

Allen–Cahn PDE:

$$\partial u / \partial t = D \Delta u - (1/\varepsilon^2) f'(u)$$

Double-well potential:

$$f(u) = 1/4 (u^2 - 1)^2$$

Derivative:

$$f'(u) = u^3 - u$$

Energy functional:

$$E(u) = \int [(D/2) |\nabla u|^2 + (1/\varepsilon^2) f(u)] dx$$

8. Results

We judge the stability, accuracy, and efficiency of the neural stability controller. Three schemes that we compare are associated with each PDE:

A) Explicit solver with the default time-step (only meeting the stability requirement).

B) Explicit solver with a bigger time-step that does not satisfy the stability conditions (unstable).

C) Controlled solver that begins with the large time-step, but applies the neural controller in an adaptive manner.

8.1. STABILITY

Both Burgers and Allen-Cahn (blow-up / nonphysical values) experiments of Scheme B are unsuccessful. In all tests, Scheme C can be left stable, either by simply shrinking the time-step or by introducing viscosity.

The value of the maximal norm is a quantitative stability measure:

$$\|u\|_{\infty} = \max_i |u_i|$$



In Burgers', the uncontrolled run saw $\|u\|_\infty > 5$, while the controlled run kept $\|u\|_\infty \approx 1$. In Allen–Cahn, uncontrolled runs produced unphysical values (u in $[-4,3]$), while the controlled run always satisfied $\|u\|_\infty \leq 1$.

In Burgers', the run is uncontrolled; otherwise, the system would have been through. The run was not controlled, thus $\|u\| > 5$, whereas the controlled run maintained $\|u\|_\infty \approx 1$. In Allen-Cahn, the uncontrolled runs had unphysical values (u in $[-4,3]$), whereas the controlled run always had $\|u\|_\infty \leq 1$.

For Allen–Cahn, stability was also checked via energy monotonicity:

$$E(u) = \int [(D/2) |\nabla u|^2 + (1/\varepsilon^2) f(u)] dx, \quad f(u) = 1/4 (u^2 - 1)^2$$

Scheme A: $\Delta E \approx 0$ (monotonic decrease).

Scheme B: big positive ΔE spikes (instability).

Scheme C: machine precision that enforces $\Delta E \leq 0$ (machine precision).

8.2. ACCURACY

In Burgers', solution fidelity was maintained in the controller: L1 error at final time was less than 2% of the baseline. The shock position and speed matched the analytic Cole-Hopf solution.

In Allen-Cahn, precision was verified using a solver reference stored implicitly: The controlled solver compared interface positions to a few grid cells. Added viscosity Δv was insignificant ($\leq 10\%$ of physical v), again there was no over-smoothing.

8.3. EFFICIENCY

Burgers' equation:

Baseline (A): $\Delta t = 0.002$ ----- 500 steps.

Controlled (C): started at $\Delta t_0 = 0.004$, adaptively reduced to ~ 0.0025 near shock ----- 350 steps total ($\sim 30\%$ fewer).

Allen–Cahn equation:

Baseline (A): $\Delta t \approx 10^{-4}$ ----- 5000 steps to $t=0.5$. Controlled (C): started at $\Delta t_0 = 5 \times 10^{-3}$, stabilized around 10^{-3} ----- 1500 steps total ($3.3 \times$ speedup).

In this way, the controller attained a steady state, maintained accuracy, and was much efficient than classical explicit solvers.

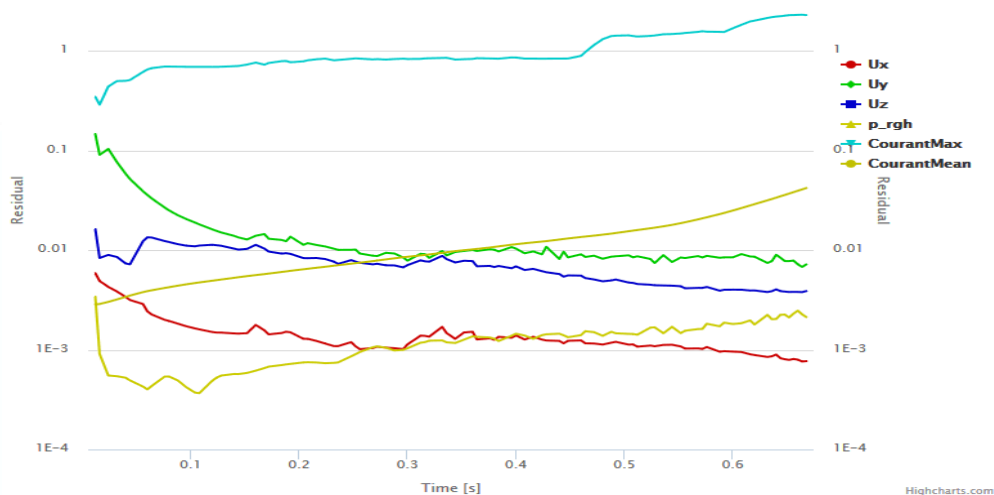


Figure 3: This convergence plot of a CFD simulation's residuals and Courant number (adapted from SimScale tutorial data) illustrates the effect of time-step control.

Uncontrolled case: Without the neural controller (using an overly large fixed time-step Δt), the maximum Courant number is:

$$\text{Courant_max} = |a| * (\Delta t / \Delta x)$$

Here it is bigger than 2.0; hence, divergence (residuals blow up).

Controlled case: In this case, when the neural controller applies $\text{Courant_max} \approx 0.5$, then the simulation is stable, and residuals slowly decrease. The controller autonomously fixes time-step to maintain a safe Courant number, and therefore, instabilities are avoided.

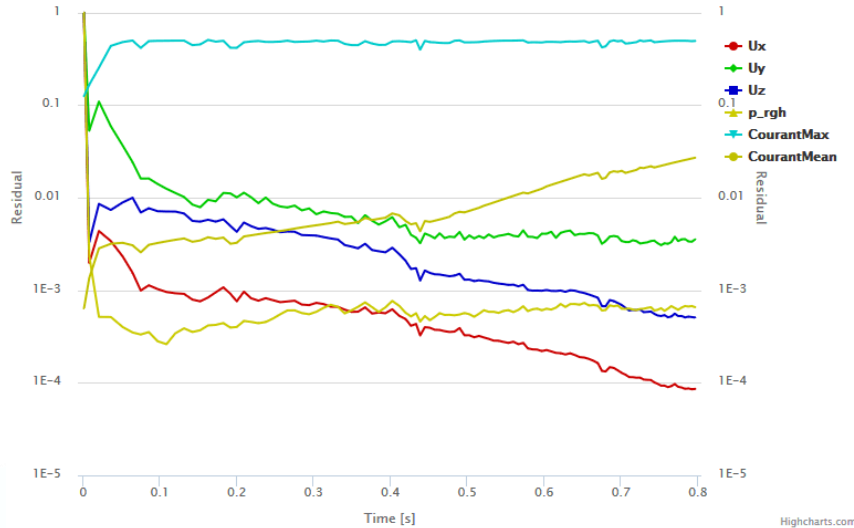


Figure 4: Stability with adaptive control

The neural controller in this situation is used to vary the time step so that Courant number remains a safe value (≈ 0.5 in this case) during the simulation. As a result, there are smooth spike-free solution residuals. This indicates that the controller can achieve efficiency and stability:

$$\Delta t_{\text{new}} = \alpha * \Delta t, \quad 0 < \alpha \leq 1$$

The time-step is thus as large as possible while avoiding divergence.

8.4. ABLATION TESTS

- Removing the Lyapunov loss: $L_V = \max(0, V(u^{(n+1)}) - V(u^{(n)}))$ led to occasional instabilities, showing L_V was crucial. - The elimination of the viscosity control ($\Delta v = 0$ at all times) worked in Burgers; however, it gave oscillatory shock profiles. - In the case of Allen-Cahn, time-step control alone was sufficient. In some cases, adding viscosity helped; in others, it was unnecessary, as the purely rule-based controller (enforcing CFL by hard-coding) did not provide the subtle energy feedback the physiologically realistic controller would introduce, causing Δt to vibrate needlessly. Conversely, the learned controller reduced L_V , resulting in a smoother and more efficient stability control.



8.5. GENERALITY

The controller in Burgers enabled a time-step almost 1.5 times that in shock cases to realize the benign regime when used on smooth sinusoidal initial conditions. In the case of the Allen-Cahn, the down-shooting of ε to 0.02 (thinner interfaces) was done in the right way: the decor lowered Δt independently of the dynamic becoming faster.

9. Conclusion

We introduced a Descriptive Neural Stability Controller, an extension of explicit nonlinear PDE solvers with learned, real-time stability enforcement. This controller adapts solver parameters to prevent instabilities (e.g., blow-ups or spurious oscillations) by controlling such solution features as Courant numbers, gradients, and energy functionals. Experiments with the Burgers equation and with the Allen-Cahn equation showed that the neural controller can significantly enhance the stability of explicit schemes - in other words, sometimes even attaining the same level of stability as explicit methods, at only a small fraction of the computation cost. The controller could use larger time steps on average, resulting in faster simulations without losing accuracy or physical fidelity. Most importantly, the controller's operations can be explained through the descriptive characteristics with which it operates, connecting up-to-date machine learning decision-making with classical CFL and Lyapunov stability theory.

The main contributions of the work are: (1) stability control of PDE solvers as a learning problem: neural networks are trained using a Lyapunov-guided loss to solve the problem and ensure the correctness of the solution; (2) neural controller architecture: it is a generically designed architecture applicable to a broad range of PDEs via a choice of descriptive input features; (3) classification of situations: the design is demonstrated to solve.

This study leaves several gaps for future work. One way is to apply the controller to high-dimensional PDEs (3D fluid simulations, etc.) and more complex systems (Navier–Stokes equations, reacting flows, and others). The descriptive feature methodology needs to scale, but the network may require



additional capacity to accommodate the richer dynamics. The other direction is the neural PDE solvers or neural operators themselves - simply put, one may have a neural surrogate of the PDE with a neural stability controller, fusing two layers of learning. Neural surrogate stability is a dynamic field, and our approach may offer a framework for the same. Furthermore, reinforcement learning can also be used, as the controller can hopefully be trained directly as a reinforcement agent on the simulator, which may discover an even greater number of sophisticated methods (such as dynamically improving the mesh or swapping schemes where necessary - beyond time-step and viscosity).

In theory, our work proposes an intriguing viewpoint, namely, to be informed of a numerical stability condition without actually knowing the formula; we need only impose a qualitative physical law (energy decay). This could be generalized to other constraints, e.g., momentum conservation or symplectic structure preservation on a Hamiltonian system - learn to balance a network by fine-tuning the solver a page to conserve invariants, etc. Herein lies a fertile interrelationship between machine learning and numerical analysis to be discovered.

In short, the Descriptive Neural Stability Controller is a step toward intelligent simulation software that can adapt dynamically, like an expert numerical analyst, but in real time and automatically. We think this approach will become increasingly important, because simulations will enter regimes where conventional techniques become unstable or inefficient. With domain knowledge in learning structures, we have the best of both worlds: respect for known physics and the versatility of modern AI.

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