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متباينات هيرميت–هادامارد وفيجير وجنسن المتقطعة للدوال الضبابية preinvex–h باستعمال تكامل شوكيه

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المستخلص:

المقدمة: تمثل متباينات هيرميت–هادامارد أداة أساسية في دراسة الدوال المحدبة والدوال المحدبة المعممة، إلا أن نقلها إلى الدوال ذات القيم الضبابية تحت قياس غير جمعي ليس مباشرًا، لأن تكامل شوكيه يتميز بالرتابة والتجانس الموجب ولا يمتلك خاصية الجمعية بصورة عامة.

طرق العمل: يعتمد البحث على تحليل المستويات للدوال الضبابية، وترتيب كوليش–ميرنكر على كل مستوى، ومسارات invex التي تحقق الشرط C، مع استعمال سعات symmetric normalized submodular. وبذلك يتحول كل احتواء ضبابي إلى زوج من المتباينات العددية لتكامل شوكيه عند كل مستوى.

النتائج: أُثبتت متباينات جديدة من نوع هيرميت–هادامارد، ومتباينة موزونة من نوع هيرميت–هادامارد–فيجير، وتقدير متقطع من نوع جنسن لتكامل شوكيه للدوال غير السالبة ذات القيم الضبابية من نوع preinvex–h. كما صيغت الثوابت بدلالة تكامل شوكيه للدالة h وللدالة الوزنية، وتختزل هذه الثوابت إلى الثوابت الكلاسيكية عند الرجوع إلى قياس ليبغ الجمعي.

الاستنتاجات: توفر النتائج إطارًا غير جمعي متماسكًا يربط التحدب الضبابي المعمم بالتكامل غير الجمعي مع تجنب افتراضات الجمعية غير الصحيحة في تكامل شوكيه.

الكلمات المفتاحية: تكامل شوكيه؛ عدد ضبابي؛ دالة preinvex–h؛ متباينة هيرميت–هادامارد؛ متباينة فيجير؛ السعة؛ قياس غير جمعي.

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Sharp Hermite-Hadamard, Fejér and Discrete Jensen-Type Inequalities for Fuzzy-Number-Valued h-Preinvex Mappings via the Choquet Integral

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ABSTRACT

Background: Hermite-Hadamard inequalities are central estimates for convex and generalized convex mappings. Their extension to fuzzy-number-valued mappings under a non-additive capacity is non-trivial because the Choquet integral is monotone and positively homogeneous but not additive in general.

Materials and Methods: This paper develops a level-wise approach based on fuzzy numbers, the Kulisch-Miranker order on all level sets, invex paths satisfying Condition C, and normalized symmetric submodular capacities. A level-wise Choquet integral is used so that every fuzzy inequality is reduced to two scalar Choquet inequalities at each level.

Results: New capacity-stable Hermite-Hadamard bounds, a weighted Hermite-Hadamard-Fejér estimate, and a discrete Choquet-Jensen-type upper estimate are established for nonnegative fuzzy-number-valued h-preinvex mappings. The constants are expressed through Choquet integrals of h and of the weight function; in the additive case they reduce to the classical integral constants. Equality is obtained for affine fuzzy-number-valued mappings when $h(t) = t$ and the capacity is Lebesgue additive.

Conclusion: The results unify fuzzy-number-valued convexity, preinvexity, and non-additive integration in a framework that avoids the invalid additivity assumptions often associated with Choquet-type inequalities.

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Keywords: Choquet integral; fuzzy number; h-preinvex mapping; Hermite-Hadamard inequality; Fejér inequality; capacity; non-additive measure.

1. INTRODUCTION

Convexity-based integral inequalities continue to be a productive source of estimates in analysis, approximation theory, optimization, and decision modelling. The classical Hermite-Hadamard inequality provides a two-sided estimate of the integral mean of a convex function using its midpoint and endpoint values. During the last decades, the inequality has been extended to several generalized convexities, including preinvex, h-convex, harmonically convex, interval-valued, and fuzzy-number-valued settings.

A second line of development replaces additive measures by capacities. Choquet introduced the theory of capacities in potential theory (Choquet, 1954), and the Choquet integral subsequently became one of the main non-additive integrals used in fuzzy measure theory and decision analysis. Murofushi and Sugeno provided an influential interpretation of fuzzy measures and the Choquet integral as an integration theory with respect to non-additive fuzzy measures (Murofushi & Sugeno, 1989). In contrast with the Lebesgue integral, the Choquet integral is generally not additive. Therefore, direct translations of classical Hermite-Hadamard proofs can be wrong unless one imposes sufficient structural assumptions, such as submodularity or strong subadditivity of the underlying capacity (Chateauneuf & Cornet, 2018).

The fuzzy-valued direction is also well established. Gong, Chen and Duan developed Choquet integrals of fuzzy-number-valued functions and characterized their level-wise behaviour with respect to non-additive measures (Gong et al., 2014). Wang showed that Hadamard-type inequalities for Choquet integrals require care: for r-convex real functions, the expected Hadamard inequality need not hold generally, while meaningful estimates can be recovered under distorted Lebesgue measures, monotonicity and submodularity assumptions (Wang, 2019). This observation is a critical starting point for the present work.

Recent studies have expanded Hermite-Hadamard inequalities to interval-valued and fuzzy-number-valued contexts. Costa and Román-Flores (2017) investigated integral inequalities for fuzzy-interval-valued functions. Zhou et al. (2022) studied interval-valued exponential-type preinvex functions through Riemann-Liouville fractional integrals, while Tan et al. (2023) established Hermite-Hadamard-type inequalities for h-preinvex interval-valued functions in a fractional setting. Khan, Zaini, Macías-Díaz, et al. (2022) introduced up-and-down h-preinvex fuzzy-number-valued mappings and derived fuzzy Riemann integral inequalities. Coordinated non-convex fuzzy-number-valued mappings and their Aumann-integral inequalities were considered by Rakhmangulov et al. (2024). Macías-Díaz et al. (2022) developed Hermite-Hadamard inequalities for generalized convex functions in interval-valued calculus. Related developments include Khan et al. (2023), Sharma et al. (2021), and Sun (2020).

The present paper is positioned at the intersection of these streams. The main novelty is not merely replacing a symbol in a known inequality. The work combines four ingredients that are not simultaneously present in the cited literature: fuzzy-number-valued mappings, h-preinvexity on invex paths, level-wise Choquet integration, and normalized symmetric submodular capacities. The submodularity assumption is not cosmetic; it is the mechanism that allows a valid subadditive estimate for the Choquet integral and prevents invalid use of linearity.

The contributions are as follows.

1. A level-wise class of nonnegative fuzzy-number-valued h-preinvex mappings is formulated with respect to an invex map satisfying Condition C.
2. A capacity-stable Hermite-Hadamard inequality is proved for the Choquet mean along an invex segment.
3. A weighted Hermite-Hadamard-Fejér inequality is obtained under a symmetric weight.

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4. A discrete Choquet-Jensen-type upper estimate is established for finite capacities.
5. Equality cases and special reductions are derived, including the additive Lebesgue case, the real-valued case, the interval-valued case, and the ordinary convex case.
6. Numerical and decision-modelling examples are provided to verify the bounds and to show how non-additive aggregation may be used without violating the mathematical assumptions.

Table 1. Comparison of selected studies with the present contribution.

Selected study	Mapping/value class	Integration framework	Scope and distinction from the present study
Gong et al. (2014)	Fuzzy-number-valued	Choquet integral	Develops fuzzy-valued Choquet integration; no h-preinvex Hermite-Hadamard theory.
Wang (2019)	Real-valued r-convex	Choquet integral	Scalar setting; no level-wise fuzzy or preinvex treatment.
Zhou et al. (2022)	Interval-valued preinvex	Riemann-Liouville fractional integral	Uses an additive fractional framework rather than a capacity-based Choquet model.
Tan et al. (2023)	h-preinvex interval-valued	Riemann-Liouville fractional integral	Does not treat fuzzy numbers or a non-additive capacity.
Khan, Zaini, Macías-Díaz, et al. (2022)	h-preinvex fuzzy-number-valued	Fuzzy Riemann integral	Does not involve a Choquet capacity or submodularity.
Rakhmangulov et al. (2024)	Coordinated fuzzy-number-valued	Fuzzy Aumann integral	Uses a different integral and a coordinated mapping setting.
Macías-Díaz et al. (2022)	Generalized convex interval-valued	Interval-valued integral calculus	No fuzzy-number-valued h-preinvex Choquet-Fejér framework.
Present study	**Fuzzy-number-valued h-preinvex**	**Choquet integral under a symmetric submodular capacity**	**Provides level-wise Hermite-Hadamard, Fejér, and discrete Jensen-type bounds under valid non-additive assumptions.**

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٢٠٢٦م/١٤٤٨هـ

2. MATERIALS AND METHODS

2.1. FUZZY NUMBERS, ARITHMETIC AND ORDER

Let Eplus denote the class of nonnegative fuzzy numbers on R whose level sets are nonempty compact intervals contained in the nonnegative real line. For U in Eplus and $0 \leq \alpha \leq 1$, the α-level interval is represented by its left and right endpoint functions U_α^L and U_α^R, where

$$0 \leq U_\alpha^L \leq U_\alpha^R. \quad (1)$$

For U and V in Eplus and $\lambda \geq 0$, fuzzy addition and scalar multiplication are understood level-wise:

$$(U \oplus V)_\alpha^L = U_\alpha^L + V_\alpha^L, \quad (2a)$$

$$(U \oplus V)_\alpha^R = U_\alpha^R + V_\alpha^R, \quad (2b)$$

$$(\lambda \odot U)_\alpha^L = \lambda U_\alpha^L, \qquad (\lambda \odot U)_\alpha^R = \lambda U_\alpha^R. \quad (2c)$$

The order relation used in the paper is the level-wise Kulisch-Miranker order:

$$U \preccurlyeq V \quad \Leftrightarrow \quad U_\alpha^L \leq V_\alpha^L \text{ and } U_\alpha^R \leq V_\alpha^R \quad \text{for all } 0 \leq \alpha \leq 1. \quad (3)$$

This order is strong enough to convert each fuzzy inclusion into two scalar inequalities. It also prevents the ambiguity that would arise from comparing fuzzy numbers only by a ranking index.

2.2. CAPACITIES AND THE CHOQUET INTEGRAL

Let $X = [0,1]$ and let $\mathcal{B}$ be its Borel σ-algebra. A capacity is a set function $\nu: \mathcal{B} \to [0,1]$ satisfying

$$\nu(\emptyset) = 0, \qquad A \subseteq B \Rightarrow \nu(A) \leq \nu(B), \qquad \nu(X) = 1. \quad (4)$$

The capacity ν is called submodular if

$$\nu(A \cup B) + \nu(A \cap B) \leq \nu(A) + \nu(B), \qquad A, B \in \mathcal{B}. \quad (5)$$

It is called symmetric with respect to the reflection $R(t) = 1 - t$ if

$$\nu(RA) = \nu(A), \qquad RA = \{1 - t : t \in A\}. \quad (6)$$

For a nonnegative measurable scalar function f on X, its Choquet integral with respect to ν is

$$(C)\int_0^1 f(t)\, d\nu(t) = \int_0^\infty \nu(\{t \in X : f(t) \geq s\})\, ds. \quad (7)$$

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العدد السادس والأربعون
٢٠٢٦م /١٤٤٨هـ

For a fuzzy-number-valued mapping F on X, write $J = (C)\int_0^1 F(t)\,dv(t)$. Its endpoint functions are defined level-wise by

$$J_\alpha^L = (C)\int_0^1 F_\alpha^L(t)\,dv(t), \quad (8a)$$

$$J_\alpha^R = (C)\int_0^1 F_\alpha^R(t)\,dv(t). \quad (8b)$$

Since $F_\alpha^L(t) \le F_\alpha^R(t)$ and the Choquet integral is monotone, the pair of endpoint functions in (8a) and (8b) is again ordered. The non-additivity of (7) is the reason for assuming submodularity when estimates require an upper bound of the integral of a sum.

2.3. INVEX PATHS AND H-PREINVEX FUZZY-NUMBER-VALUED MAPPINGS

Let $K \subseteq R$ and let $\eta: K \times K \to R$ be a mapping. The set K is invex with respect to η if

$$x + t\eta(y,x) \in K, \qquad x, y \in K, \quad 0 \le t \le 1. \quad (9)$$

The invex map η is said to satisfy Condition C if, for all $x, y \in K$ and $t \in [0,1]$,

$$\eta\big(y, y + t\eta(x,y)\big) = -t\eta(x,y), \qquad \eta\big(x, y + t\eta(x,y)\big) = (1-t)\eta(x,y). \quad (10)$$

Condition C implies the useful identity

$$\eta\big(x + t_2\eta(y,x), x + t_1\eta(y,x)\big) = (t_2 - t_1)\eta(y,x), \quad 0 \le t_1, t_2 \le 1. \quad (11)$$

Let h be a nonnegative measurable function on $[0,1]$ and suppose that $h(1/2) > 0$. A mapping F on K is called fuzzy-number-valued h-preinvex with respect to η if

$$F\big(x + t\eta(y,x)\big) \preccurlyeq h(1-t) \odot F(x) \oplus h(t) \odot F(y) \quad (12)$$

for all $x, y \in K$ and $t \in [0,1]$.

2.4. METHODOLOGICAL VALIDITY CONDITIONS

All main results below use the following assumptions.

Assumption A. K is invex with respect to η, and η satisfies Condition C.

Assumption B. F on K is fuzzy-number-valued h-preinvex, and for every $\alpha \in [0,1]$ the endpoint functions along the path $z_t = x + t\eta(y,x)$ are measurable and Choquet integrable.

Assumption C. ν is a normalized, symmetric, and submodular capacity on $[0,1]$.

The proof strategy is level-wise. Each fuzzy inequality is established by proving the corresponding left and right endpoint inequalities for every $\alpha \in [0,1]$, then returning to the fuzzy order (3).

3. RESULTS AND DISCUSSION

3.1. AUXILIARY LEMMAS

Lemma 1. Let ν be a normalized submodular capacity. If fuzzy-number-valued mappings F and G on $[0,1]$ are level-wise Choquet integrable and $a, b \geq 0$, then:

$$F(t) \preccurlyeq G(t) \text{ for all } t \Rightarrow (C)\int_0^1 F\, d\nu \preccurlyeq (C)\int_0^1 G\, d\nu, \quad (13)$$

and

$$(C)\int_0^1 (a \odot F \oplus b \odot G)\, d\nu$$

$$\preccurlyeq a \odot (C)\int_0^1 F\, d\nu \oplus b \odot (C)\int_0^1 G\, d\nu. \quad (14)$$

If, in addition, ν is symmetric, then

$$(C)\int_0^1 F\,(1-t)\, d\nu(t) = (C)\int_0^1 F\,(t)\, d\nu(t). \quad (15)$$

Proof. Fix $\alpha \in [0,1]$ and choose $S \in \{L,R\}$. If $F(t) \preccurlyeq G(t)$, then, at level α, the corresponding S-endpoints are ordered for every t. Scalar monotonicity of the Choquet integral therefore gives the S-endpoint inequality in (13).

For (14), take the S-endpoint of $a \odot F \oplus b \odot G$. Positive homogeneity first extracts the nonnegative scalars a and b, and subadditivity under the

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العدد السادس والأربعون
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submodular capacity then separates the two scalar endpoint functions. Repeating this argument for S=L and S=R yields the fuzzy inequality in (14). Finally, under the reflection that sends t to 1-t, the level sets in the definition (7) are reflected subsets of [0,1]. The identity ν(RA)=ν(A) therefore gives (15) at both endpoints. The Kulisch-Miranker order (3) completes the proof. □

Lemma 2. If F is fuzzy-number-valued h-preinvex, then for every $\alpha \in [0,1]$ the left and right endpoint functions F_α^L and F_α^R are scalar h-preinvex along each invex path, namely

$$F_\alpha^S\big(x + t\eta(y,x)\big) \le h(1-t)F_\alpha^S(x) + h(t)F_\alpha^S(y), \quad S = L, R. \quad (16)$$

Proof. Taking the α-level sets on both sides of (12) and using (2) gives the two scalar inequalities in (16). ▫

3.2. MAIN HERMITE-HADAMARD INEQUALITY

For $x, y \in K$, define the Choquet mean along the invex path by

$$\mathcal{C}_\nu(F; x, y) = (C)\int_0^1 F\left(x + t\eta(y,x)\right) d\nu(t). \quad (17)$$

Also put

$$\beta_h^L = (C)\int_0^1 h\,(1-t)\,d\nu(t). \quad (18a)$$

$$\beta_h^R = (C)\int_0^1 h\,(t)\,d\nu(t). \quad (18b)$$

Theorem 1. Under Assumptions A-C, for all $x, y \in K$,

$$\frac{1}{2h(1/2)} \odot F\left(x + \frac{1}{2}\eta(y,x)\right) \preccurlyeq \mathcal{C}_\nu(F; x, y)$$
$$\preccurlyeq \beta_h^L \odot F(x) \oplus \beta_h^R \odot F(y). \quad (19)$$

If ν is symmetric, then $\beta_h^L = \beta_h^R = \beta_h$ and (19) becomes

$$\frac{1}{2h(1/2)} \odot F\left(x + \frac{1}{2}\eta(y,x)\right) \preccurlyeq \mathcal{C}_\nu(F; x, y)$$
$$\preccurlyeq \beta_h \odot \big(F(x) \oplus F(y)\big). \quad (20)$$

Proof. Write $z_t = x + t\eta(y,x)$. By (11),

$$z_{1/2} = z_t + \frac{1}{2}\eta(z_{1-t}, z_t). \quad (21)$$

Using h-preinvexity with the pair (z_t, z_{1-t}) and parameter $1/2$ gives

$$F(z_{1/2}) \preccurlyeq h(1/2) \odot F(z_t) \oplus h(1/2) \odot F(z_{1-t}). \quad (22)$$

Fix α ∈ [0,1] and S ∈ {L,R}. Taking the S-endpoint of (22) gives a scalar inequality. Because ν is normalized, the Choquet integral of the constant midpoint endpoint equals that endpoint. Monotonicity permits integration of the right-hand side, positive homogeneity extracts h (1/2), and subadditivity separates the two endpoint functions. Consequently,

$$F(z_{1/2}) \preccurlyeq h(1/2) \odot \left((C)\int_0^1 F\,(z_t)\,dv(t) \oplus (C)\int_0^1 F\,(z_{1-t})\,dv(t) \right). \quad (23)$$

By the reflection identity (15), the two Choquet integrals in (23) coincide and both equal the mean defined in (17). Thus (23) contains two identical terms. Since h(1/2)>0, division by 2h(1/2) yields

$$\frac{1}{2h(1/2)} \odot F(z_{1/2}) \preccurlyeq (C)\int_0^1 F\,(z_t)\,dv(t). \quad (24)$$

For the upper estimate, apply h-preinvexity directly to the endpoints x and y along the invex path introduced in (17). Equation (12) gives

$$F(z_t) \preccurlyeq h(1-t) \odot F(x) \oplus h(t) \odot F(y). \quad (25)$$

Fix α and S once more. Monotonicity applied to the S-endpoint of (25) produces the Choquet integral of the sum of the two endpoint terms. Subadditivity separates that sum, and positive homogeneity factors out the nonnegative endpoint constants at x and y. By the definitions (18a)-(18b), this gives

$$(C)\int_0^1 F\,(z_t)\,dv(t) \preccurlyeq \beta_h^L \odot F(x) \oplus \beta_h^R \odot F(y). \quad (26)$$

Combining (24) and (26) for both choices S=L and S=R and then applying the Kulisch-Miranker order (3) proves (19). Finally, the reflection identity

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العدد السادس والأربعون
٢٠٢٦م /١٤٤٨هـ

(15) shows that the two coefficients defined in (18a) and (18b) are equal; denoting their common value as in (20) completes the proof. □

3.3. SHARPNESS AND SPECIAL CASES

Corollary 1. If ν is the Lebesgue measure λ and $h(t) = t$, then (20) reduces to

$$F\left(x + \frac{1}{2}\eta(y,x)\right) \preccurlyeq \int_0^1 F\left(x + t\eta(y,x)\right) dt \preccurlyeq \frac{1}{2} \odot \left(F(x) \oplus F(y)\right). \quad (27)$$

If F is affine on the invex path, that is,

$$F\left(x + t\eta(y,x)\right) = (1-t) \odot F(x) \oplus t \odot F(y), \quad t \in [0,1], \quad (28)$$

then equality holds on both sides of (27). Therefore the constants in (27) cannot be improved in the classical affine fuzzy case.

Corollary 2. If F is real-valued, then (19) gives the scalar Choquet Hermite-Hadamard inequality for nonnegative h-preinvex functions. If F is interval-valued, (19) gives the corresponding interval-valued inclusion under the Kulisch-Miranker order.

Corollary 3. If $\eta(y,x) = y - x$, then K is an ordinary convex interval and (19) becomes a Choquet Hermite-Hadamard inequality for fuzzy-number-valued h-convex mappings.

3.4. WEIGHTED HERMITE-HADAMARD-FEJÉR INEQUALITY

Let w be a nonnegative measurable function on $[0,1]$, Choquet integrable, and symmetric in the sense that

$$w(t) = w(1-t), \qquad t \in [0,1]. \quad (29)$$

Define

$$\gamma_w = (C)\int_0^1 w(t)\, d\nu(t). \quad (30a)$$

$$\gamma_{h,w}^L = (C)\int_0^1 w(t) h(1-t)\, d\nu(t). \quad (30b)$$

$$\gamma_{h,w}^R = (C)\int_0^1 w(t) h(t)\, d\nu(t). \quad (30c)$$

Theorem 2. Under Assumptions A-C and (29),

JOBS مجلة العلوم الأساسية
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العدد السادس والأربعون
٢٠٢٦م /١٤٤٨هـ

$$\frac{\gamma_w}{2h(1/2)} \odot F\left(x+\frac{1}{2}\eta(y,x)\right) \preccurlyeq (C)\int_0^1 w(t) \odot F\big(x+t\eta(y,x)\big)\,dv(t)$$
$$\preccurlyeq \gamma_{h,w}^{L} \odot F(x) \oplus \gamma_{h,w}^{R} \odot F(y). \quad (31)$$

Proof. Let the invex-path point be defined as in (17), fix α ∈ [0,1], and choose S ∈ {L,R}. Multiplying the S-endpoint form of (22) by the nonnegative weight w(t) preserves the inequality.

After Choquet integration, positive homogeneity converts the left-hand side into the coefficient defined in (30a) times the midpoint endpoint. Monotonicity and subadditivity bound the right-hand side by h(1/2) times the sum of the two weighted endpoint integrals.

For the reflected term, w(t)=w(1-t) by (29), and the relevant level sets are mapped into one another by reflection about the midpoint. Symmetry of ν therefore makes the two weighted Choquet integrals equal. Division by 2h (1/2) gives the lower estimate in (31).

For the upper estimate, multiply the S-endpoint form of (25) by w(t). Monotonicity, subadditivity, and positive homogeneity give the two coefficients defined in (30b)-(30c). The argument holds for both endpoints, so (3) yields the upper fuzzy inequality and completes the proof of (31). □

For $w \equiv 1$, Theorem 2 reduces to Theorem 1.

3.5. DISCRETE CHOQUET-JENSEN-TYPE ESTIMATE

Let $N = \{1,2,\dots,n\}$ and let $\mu: 2^N \to [0,1]$ be a normalized submodular finite capacity. For a nonnegative vector $a = (a_1,\dots,a_n)$, the discrete Choquet integral is

$$(C)\sum_{i=1}^{n} a_i\ d\mu = \sum_{k=1}^{n}\big(a_{(k)} - a_{(k-1)}\big)\,\mu(\{(k),(k+1),\dots,(n)\}), \quad a_{(0)} = 0, \quad (32)$$

where $a_{(1)} \le \cdots \le a_{(n)}$ is the increasing rearrangement.

Theorem 3. Let $t_1,\dots,t_n \in [0,1]$ and $z_i = x + t_i\eta(y,x)$. If F is fuzzy-number-valued h-preinvex, then

JOBS مجلة العلوم الأساسية Journal of Basic Science
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العدد السادس والأربعون
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$$(C)\sum_{i=1}^{n} F\,(z_i)\,d\mu$$
$$\preccurlyeq \left((C)\sum_{i=1}^{n} h\,(1-t_i)\,d\mu\right) \odot F(x) \oplus \left((C)\sum_{i=1}^{n} h\,(t_i)\,d\mu\right)$$
$$\odot F(y). \quad (33)$$

Proof. For each i, (12) gives

$$F(z_i) \preccurlyeq h(1-t_i) \odot F(x) \oplus h(t_i) \odot F(y). \quad (34)$$

Fix α ∈ [0,1] and S ∈ {L,R}. At this level and endpoint, denote the scalar sequence on the left side of (34) by a, and the two scalar sequences on its right side by b and c. The endpoint form of (34) states componentwise that each entry of a does not exceed the corresponding entry of b+c.

Monotonicity first bounds the finite Choquet integral of a by that of b+c. Because μ is submodular, subadditivity separates this latter integral into the integrals of b and c. Positive homogeneity then factors out the endpoint constants at x and y, leaving exactly the two discrete coefficients displayed in (33). Applying the argument for both endpoints and using (3) proves (33).
□

If μ is additive with weights $p_i \geq 0$ and $\sum_{i=1}^{n} p_i = 1$, (33) becomes

$$\sum_{i=1}^{n} p_i \odot F(z_i)$$
$$\preccurlyeq \left(\sum_{i=1}^{n} p_i\, h(1-t_i)\right) \odot F(x) \oplus \left(\sum_{i=1}^{n} p_i\, h(t_i)\right)$$
$$\odot F(y). \quad (35)$$

For $h(t) = t$, the upper side of (35) is the classical affine Jensen-type endpoint estimate.

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4. NUMERICAL VALIDATION

4.1. CONTINUOUS EXAMPLE WITH DISTORTED LEBESGUE CAPACITY

Let $K=[0,1]$, $\eta(y,x)=y-x$, $x=0$, $y=1$, and $h(t)=t$. Consider the triangular fuzzy-number-valued mapping

$$F(u)=(u^2,u^2+0.1,u^2+0.2),\qquad 0\le u\le 1. \quad (36)$$

Its endpoint functions are

$$F_\alpha^L(u)=u^2+0.1\alpha,\qquad F_\alpha^R(u)=u^2+0.2-0.1\alpha. \quad (37)$$

Both endpoint functions are convex, hence F is fuzzy-number-valued convex and therefore h-preinvex for $h(t)=t$.

Let $\nu_q(A)=\lambda(A)^q$ with $q=1/2$, where λ denotes Lebesgue measure. This capacity is normalized, symmetric and submodular. For $f(t)=t^2$,

$$(C)\int_0^1 t^2\, d\nu_q(t)=\int_0^1 \left(1-\sqrt{s}\right)^q\, ds=\frac{2}{(q+1)(q+2)}=0.5333. \quad (38)$$

For $h(t)=t$,

$$\beta_h=(C)\int_0^1 t\, d\nu_q(t)=\int_0^1 (1-s)^q\, ds=\frac{1}{q+1}=0.6667. \quad (39)$$

Therefore, for $\alpha\in[0,1]$,

$$M_\alpha^L=0.5333+0.1\alpha,\qquad M_\alpha^R=0.7333-0.1\alpha. \quad (40)$$

The midpoint and upper bounds in Theorem 1 are

$$B_\alpha^L=0.25+0.1\alpha,\qquad B_\alpha^R=0.45-0.1\alpha. \quad (41)$$

and

$$T_\alpha^L=0.6667+0.1333\alpha,\qquad T_\alpha^R=0.9333-0.1333\alpha. \quad (42)$$

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Table 2. Level-wise verification of Theorem 1 for the triangular fuzzy example.

α	Lower endpoint pair	Choquet endpoint pair	Upper endpoint pair
0.0	[0.2500, 0.4500]	[0.5333, 0.7333]	[0.6667, 0.9333]
0.5	[0.3000, 0.4000]	[0.5833, 0.6833]	[0.7333, 0.8667]
1.0	[0.3500, 0.3500]	[0.6333, 0.6333]	[0.8000, 0.8000]

The table verifies the inclusion

$$B_\alpha^L \le M_\alpha^L \le T_\alpha^L, \qquad B_\alpha^R \le M_\alpha^R \le T_\alpha^R. \quad (43)$$

at representative levels. Because all expressions are affine in α, the inclusion holds for all $\alpha \in [0,1]$.

4.2. FINITE CHOQUET AGGREGATION EXAMPLE

The finite version (33) is useful in fuzzy decision aggregation. Consider three criteria C1, C2 and C3 and the normalized submodular capacity

$$\mu(\{1\}) = 0.60, \quad \mu(\{2\}) = 0.55, \quad \mu(\{3\}) = 0.50. \quad (44a)$$

$$\mu(\{1,2\}) = 0.82, \quad \mu(\{1,3\}) = 0.80, \quad \mu(\{2,3\}) = 0.78, \quad \mu(N) = 1. \quad (44b)$$

The submodularity inequalities (5) hold for all pairs of subsets of N. Suppose three alternatives are assessed by triangular fuzzy ratings as in Table 3.

Table 3. Triangular fuzzy ratings for a three-criterion aggregation example.

Alternative	C1	C2	C3
A	(0.72, 0.78, 0.84)	(0.68, 0.74, 0.80)	(0.55, 0.62, 0.70)
B	(0.62, 0.70, 0.77)	(0.76, 0.82, 0.88)	(0.60, 0.66, 0.73)
C	(0.67, 0.74, 0.81)	(0.65, 0.71, 0.77)	(0.70, 0.76, 0.83)

Applying the discrete Choquet integral component-wise gives Table 4.

Table 4. Discrete Choquet aggregated triangular scores.

Alternative	Choquet aggregated score	Rank by modal value
A	(0.6806, 0.7424, 0.8060)	3
B	(0.6934, 0.7588, 0.8233)	1
C	(0.6810, 0.7440, 0.8120)	2

The example illustrates that finite Choquet aggregation can be used in a fuzzy-number-valued environment while preserving the non-additive structure of the capacity. In applications, this is relevant when criteria are redundant or substitutive rather than independent. Such decision-modelling motivation is consistent with the role of fuzzy integrals in multicriteria decision making (Grabisch, 1996).

5. COMPARISON WITH EXISTING RESULTS

The proposed estimates generalize several known directions.

First, if the fuzzy number degenerates to an ordinary scalar and $\eta(y,x) = y - x$, Theorem 1 becomes a Choquet-type Hermite-Hadamard estimate for real-valued h-convex functions. This is consistent with the scalar Choquet perspective used in (Wang, 2019), but the present theorem explicitly uses submodularity and symmetry to control the lower and upper bounds.

Second, if the capacity is additive, the Choquet integral becomes the Lebesgue integral and the subadditivity step becomes equality. Hence, the present framework recovers the ordinary fuzzy-number-valued Hermite-Hadamard form for h-preinvex mappings, while retaining the level-wise fuzzy order used in recent fuzzy-valued studies (Khan, Zaini, Macías-Díaz, et al., 2022; Rakhmangulov et al., 2024).

Third, if the fuzzy number is restricted to an interval, the theorem becomes an interval-valued h-preinvex Hermite-Hadamard inequality. This places the result near the interval-valued fractional works of Zhou et al. (2022) and Tan et al. (2023), but with a different integration mechanism: the present integral is non-additive and capacity-driven rather than Riemann-Liouville fractional.

Fourth, Theorem 2 gives a Fejér-weighted version in which the weight symmetry condition $w(t) = w(1-t)$ replaces the classical symmetry about the midpoint. The proof is deliberately formulated without assuming Choquet additivity; it uses the reflection invariance of the capacity and subadditivity.

Finally, the finite estimate (33) connects the theory with discrete fuzzy aggregation. Unlike ordinary weighted means, a finite Choquet integral can represent interactions among subsets of criteria, which helps explain its continued use in multicriteria analysis (Grabisch, 1996). The universal-integral perspective also supports the interpretation of Choquet and Sugeno-type integrals within a common non-additive integration framework (Klement et al., 2010).

6. CONCLUSION

This paper established a level-wise Choquet framework for Hermite-Hadamard, Hermite-Hadamard-Fejér, and discrete Jensen-type estimates for fuzzy-number-valued h-preinvex mappings. The central mathematical point is that Choquet-type bounds require explicit structural assumptions on the capacity and cannot be justified by treating the Choquet integral as additive. Under normalization, reflection symmetry, and submodularity, the proposed method yields valid lower and upper fuzzy inclusions in the Kulisch-Miranker order. The results recover real-valued, interval-valued, additive, and ordinary convex cases under appropriate choices of F, h, η, and ν, and the numerical examples confirm the theoretical estimates.

These inequalities suggest applications in decision theory and fuzzy optimization, including bounds for Choquet-based multicriteria utilities and robustness or error analysis for non-additive objectives along invex paths. Future work may examine data-driven capacities, fractional or bipolar Choquet operators, and higher-dimensional h-preinvex models.

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