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حل الأنظمة غير الخطية بالاعتماد على تحسين خوارزمية النسر الاصلع

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المستخلص:

يُعدّ حلّ أنظمة المعادلات غير الخطية مشكلةً مهمةً في الرياضيات التطبيقية والهندسة والحوسبة العلمية. غالبًا ما تتطلب الطرق العددية التقليدية معلوماتٍ عن المشتقات وتقريباتٍ أوليةً مناسبة، مما قد يُقلّل من فعاليتها عند التعامل مع الأنظمة غير الخطية المعقدة. تقترح هذه الورقة البحثية خوارزمية بحث النسر الأصلع المُحسّنة (IBESA) لحلّ أنظمة المعادلات غير الخطية. تُحسّن الطريقة المقترحة خوارزمية بحث النسر الأصلع الأصلية (BES) من خلال دمج استراتيجية أرشفة للحفاظ على تقريبات الجذور الواعدة وآلية تنويع المجموعة لمنع التقارب المُبكر. يتم تحويل النظام غير الخطي إلى مسألة تحسين شاملة عن طريق تقليل المعيار الإقليدي لمتجه البواقي. تم تقييم أداء IBESA باستخدام ستة أنظمة غير خطية مرجعية تُمثّل نماذج متعددة الحدود، وغير قابلة للتفاضل، ومثلثية، وأسية. تم حل كل مسألة من خلال ٣٠ تجربة مستقلة، وقورنت النتائج مع خوارزميات تحسين سرب الجسيمات(BES) ، وتحسين سرب الجسيمات(PSO) ، والخوارزمية الجينية(GA) ، وخوارزمية اليراعات المعدلة (MODFA). أظهرت النتائج التجريبية أن خوارزمية

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IBESAنجحت في تحديد جميع الجذور المعروفة، وحققت نسبة نجاح ١٠٠% في جميع مسائل الاختبار. أنتجت الخوارزمية أخطاء متبقية ضئيلة للغاية، وأظهرت دقة فائقة مقارنةً بخوارزميات BESو PSOوGA، كما تفوقت على خوارزمية MODFA من حيث الدقة العددية. أكدت التحليلات الإحصائية متانة وفعالية المنهج المقترح. تشير النتائج إلى أن IBESA هي طريقة تحسين موثوقة وفعالة وخالية من المشتقات لحل الأنظمة غير الخطية.

الكلمات المفتاحية: أنظمة المعادلات غير الخطية، بحث النسر الأصلع، التحسين فوق الاستدلال، إيجاد الجذور، التحسين العالمي، IBESA.

Solving nonlinear systems based on improved bald eagle search algorithm

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Abstract

Solving systems of nonlinear equations is an important problem in applied mathematics, engineering, and scientific computing. Traditional numerical methods often require derivative information and suitable initial approximations, which may reduce their effectiveness when dealing with complex nonlinear systems. This paper proposes an Improved Bald Eagle

Search Algorithm (IBESA) for solving systems of nonlinear equations. The proposed method enhances the original Bald Eagle Search (BES) algorithm by integrating an archive strategy to preserve promising root approximations and a population diversification mechanism to prevent premature convergence. The nonlinear system is transformed into a global optimization problem by minimizing the Euclidean norm of the residual vector. The performance of IBESA was evaluated using six benchmark nonlinear systems representing polynomial, non-differentiable, trigonometric, and exponential models. Each problem was solved through 30 independent runs and compared with BES, Particle Swarm Optimization (PSO), Genetic Algorithm (GA), and Modified Firefly Algorithm (MODFA). Experimental results showed that IBESA successfully identified all known roots and achieved a 100% success rate across all test problems. The algorithm produced extremely small residual errors and demonstrated superior accuracy compared with BES, PSO, and GA, while also outperforming MODFA in terms of numerical precision. Statistical analyses further confirmed the robustness and effectiveness of the proposed approach. The results indicate that IBESA is a reliable and efficient derivative-free optimization method for solving nonlinear systems.

Keywords: Nonlinear Systems of Equations, Bald Eagle Search, Metaheuristic Optimization, Root Finding, Global Optimization, IBESA.

1. Introduction

Many sciences, including economics, engineering, chemistry, engineering, chemistry, and robotics, are associated with nonlinear equation systems (Jaberipour, Khorram, Karimi, & Applications, 2011). If the equations of the system do not have interesting linear polynomial features, the task becomes indeterminately polynomial-time difficult (Abdollahi, Isazadeh, Abdollahi, & Applications, 2013). However, such difficulties still arise when trying to solve nonlinear equation systems using the particle ensemble method, which is a combination of the conjugate direction method (CD) and Newtonian type methods, and chaotic search. Newton-type algorithms are the most popular, although their effectiveness and consistency can be greatly affected by the preliminary formulations of the solutions to the algorithms. Thus, an incorrect initial guess will cause the algorithm to fail (Luo, Tang, & Zhou, 2008). It is very important to find efficient algorithms for solving systems of nonlinear equations (Mo, Liu, Wang, & Applications, 2009). Let systems of nonlinear equations have the following form:

$$\begin{cases} f_1(x_1, x_2, \ldots, x_n) = 0 \\ f_2(x_1, x_2, \ldots, x_n) = 0 \\ \qquad \vdots \\ f_n(x_1, x_2, \ldots, x_n) = 0 \end{cases} \qquad (1)$$

The above equation can be converted into an optimization problem, we will use the helper function (Abdollahi et al., 2013):

$$\min f(x) = \sum_{i=1}^{n} f_i^2(x), \qquad x = (x_1, x_2, \ldots, x_n) \qquad (2)$$

However, choosing a suitable starting point is difficult for most nonlinear equation systems. Many teachers have developed programs to find the best answers to these questions (Tawhid & Ibrahim, 2023). The transformation of nonlinear equation systems into global minimization optimization problems is one of the well-known approaches. It is well known that there are two different methods for classical optimization: gradient search and direct search. The need to search multiple industries can slow down direct search methods; On the other hand, gradient detection methods are fast but not for less smooth discretization tasks (Abdollahi et al., 2013; Tawhid & Ibrahim, 2023).

Application of completed research work derived from first and / or second degrees, in research applied to the direct national assessment process: subjectivity and limitations only. Furthermore, all travelers seek local benefits, and so the search for local benefits nearly ignores global benefits (Ibrahim, Tawhid, & engineering, 2019).

Metaheuristic algorithms (MAs) have been widely used to solve different types of optimization problems without requiring derivative information.

This makes them particularly useful for nonlinear problems where derivatives may be difficult to obtain or conventional methods may depend strongly on the initial solution. These algorithms search for suitable solutions by exploring different regions of the search space while gradually improving the most promising solutions. Several metaheuristic approaches have been applied to nonlinear optimization problems and systems of nonlinear equations (Ibrahim et al., 2019; Tawhid & Ibrahim, 2023). In this study, an improved version of the Bald Eagle Search algorithm, called IBESA is proposed to solve systems of nonlinear equations by transforming them into optimization problems.

Convert nonlinear system of equation to optimization problem a nonlinear system can define as:

$$\begin{cases} f_1(x_1, x_2, \ldots, x_n) = 0 \\ f_2(x_1, x_2, \ldots, x_n) = 0 \\ \vdots \\ f_n(x_1, x_2, \ldots, x_n) = 0 \end{cases} \tag{3}$$

Where $f_i(x_i)$ is a function, $i = 1, 2, \ldots, n$, and n is the number of equation and variables.

The nonlinear system is reformulated as an optimization problem by defining an objective function based on the Euclidean norm of the residual vector. Accordingly, the solution is obtained by minimizing the following function:

$$\min_X \mathrm{Res}(X) = \|F(X)\|_2 = \sqrt{\sum_{i=1}^{m} [f_i(X)]^2} \tag{4}$$

After these conversions, we can solve Eq. (4) by optimization methods, so we will use the Bald Eagle Search algorithm in this paper.

2. Bald eagle search algorithm (BES)

In 2020, H.A. Alsattar (Alsattar, Zaidan, & Zaidan, 2020) proposed the Bald Eagle Search (BES) algorithm. Known for its strong cross-country

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visibility, it takes inspiration from the feeding habits of bald eagles. Three steps make up the algorithm: selecting a search area, searching within it, and swooping. Figure 1 illustrates the three stages of BES predation.

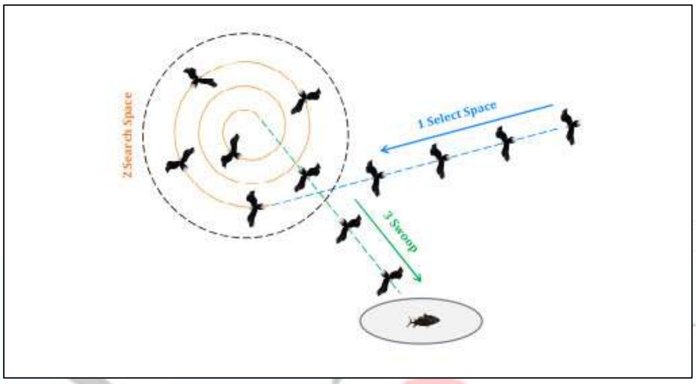

Figure 1. Graphical representation of BES captures

The bald eagle excavates the most promising position within the allocated search space to initiate the side selection technique. It affects this choice without struggling to have food in all areas. The numerical model presentation of this section is shown below (Zhang et al., 2024):

$$P_{i,new} = P_{best} + \alpha * r(P_{mean} - P_i) \quad (5)$$

Where, $P_{i,new}$ is a revised position of the i - th bald eagle,

P_{best} it shows the best position of the bald eagle at the moment,
Parameters of state change $\alpha \in (1.5, 2)$,
r is a random number belonging to (0,1),
P_{mean} is the average position of the bald eagle,
P_i denotes the i - th position of the bald eagle (Zhang et al., 2024).

Once the selected phase is completed, the BES moves to the required phase. In this phase, the algorithm mimics the hunting strategy of a bald eagle by searching for prey in a predefined area. Ultimately, lobster hunting increases circularly and expands diagonally. The numerical representation of this component can be expressed as follows:

$$P_{i,new} = P_i + y(i) * (P_i - P_{i+1}) + x(i) * (P_i - P_{mean}) \tag{6}$$

$$x(i) = \frac{xr(i)}{\max(|xr|)}, \; y(i) = \frac{yr(i)}{\max(|yr|)} \tag{7}$$

$$xr(i) = r(i) * \sin(\theta(i)), yr(i) = r(i) * \cos(\theta(i)) \tag{8}$$

$$\theta(i) = a * \pi * \text{rand} \; and \quad r(i) = \theta(i) + R * \text{rand} \tag{9}$$

Growth coefficients c_1 and c_2 are part of the BES and have values between 1 and 2.

3. Solving Systems of Nonlinear Equations Using the Improved Bald Eagle Search Algorithm (IBESA)

Solving systems of nonlinear equations is one of the most important challenges in scientific computing, engineering design, optimization, and applied mathematics. Traditional numerical methods such as Newton's method and quasi-Newton approaches often require derivative information and suitable initial guesses. Their performance may deteriorate when dealing with highly nonlinear systems, multiple roots, or complex search spaces.

To overcome these limitations, this study proposes an Improved Bald Eagle Search Algorithm (IBESA) for solving systems of nonlinear equations. The proposed method is developed based on the original Bald Eagle Search (BES) algorithm by incorporating two additional mechanisms: an archive strategy and a population diversification strategy. These modifications are intended to improve convergence accuracy, enhance exploration capability, and reduce the possibility of premature convergence.
Consider a nonlinear system represented by $F(X) = (f_1(X), f_2(X), \ldots, f_n(X)) = 0$, where $X = (x_1, x_2, \ldots, x_n)$ denotes the vector of unknown variables.
Rather than solving the nonlinear system directly, the problem is reformulated as a global optimization task. This is achieved by measuring the residual error using the Euclidean norm. For a candidate solution X, the objective function is expressed as

$$\mathrm{Res}(X) = \|F(X)\|_2 = \sqrt{\sum_{i=1}^{m}[f_i(X)]^2} \qquad (10)$$

Since this function is always non-negative, its value decreases as the candidate solution moves closer to a root of the system. Ideally, $\mathrm{Res}(X) = 0$indicates that all equations are satisfied simultaneously and that Xrepresents an exact solution of the nonlinear system.

A randomly generated population, dispersed at some level within the predefined search area, initiates the search process. Selecting, searching, and swooping are the 3 sequential steps that the IBESA algorithm offers to complete with each iteration, similar to the traditional BES method.

Collection Methods:

Candidate solutions are stored in a register structure whenever they meet a predetermined level of accuracy. Using this approach, a set of rules can preserve several original assumptions during the course of the optimization technique and can avoid missing dangerous answers.

Strategies for Population Diversity:

If the population does not exhibit sufficient diversity, a subset of the capacity response in the Seek region is randomly generated. This technique complements the ability to find a set of rules and helps to break unbound from local optima.

When solving complex search scenarios and nonlinear structures with more than one root, the set of population diversity and registration improves overall performance by maintaining a good balance between discovery and exploitation.

Table (1): Parameter Settings of the Proposed IBESA

Parameter	Symbol	Setting
Population size	N	50
Random initial solutions	N	50
Opposition-based solutions	N_{opp}	50
Total candidate solutions during initialization	$N + N_{opp}$	100
Solutions retained after initial evaluation	—	Best 50 solutions
Maximum number of iterations	$MaxIter$	3000

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Number of independent runs	—	30
Objective function	$Res(X)$	$\|F(X)\|_2 = \sqrt{\sum_{i=1}^{m}[f_i(X)]^2}$
Stopping tolerance	Tol	10^{-8}
Termination condition	—	$Res(X) \leq 10^{-8}$or $MaxIter = 3000$

The population size was fixed at 50 throughout the optimization process. At the initialization stage, 50 candidate solutions were randomly generated within the specified search bounds. For each candidate, an opposition-based solution was also generated, resulting in 100 solutions being evaluated initially. The 50 solutions with the lowest residual values were then selected to form the initial population. The algorithm was run for a maximum of 3000 iterations, and each test problem was independently executed 30 times. The search process was stopped when the residual value reached 10^{-8} or lower; otherwise, it continued until the maximum number of iterations was reached.

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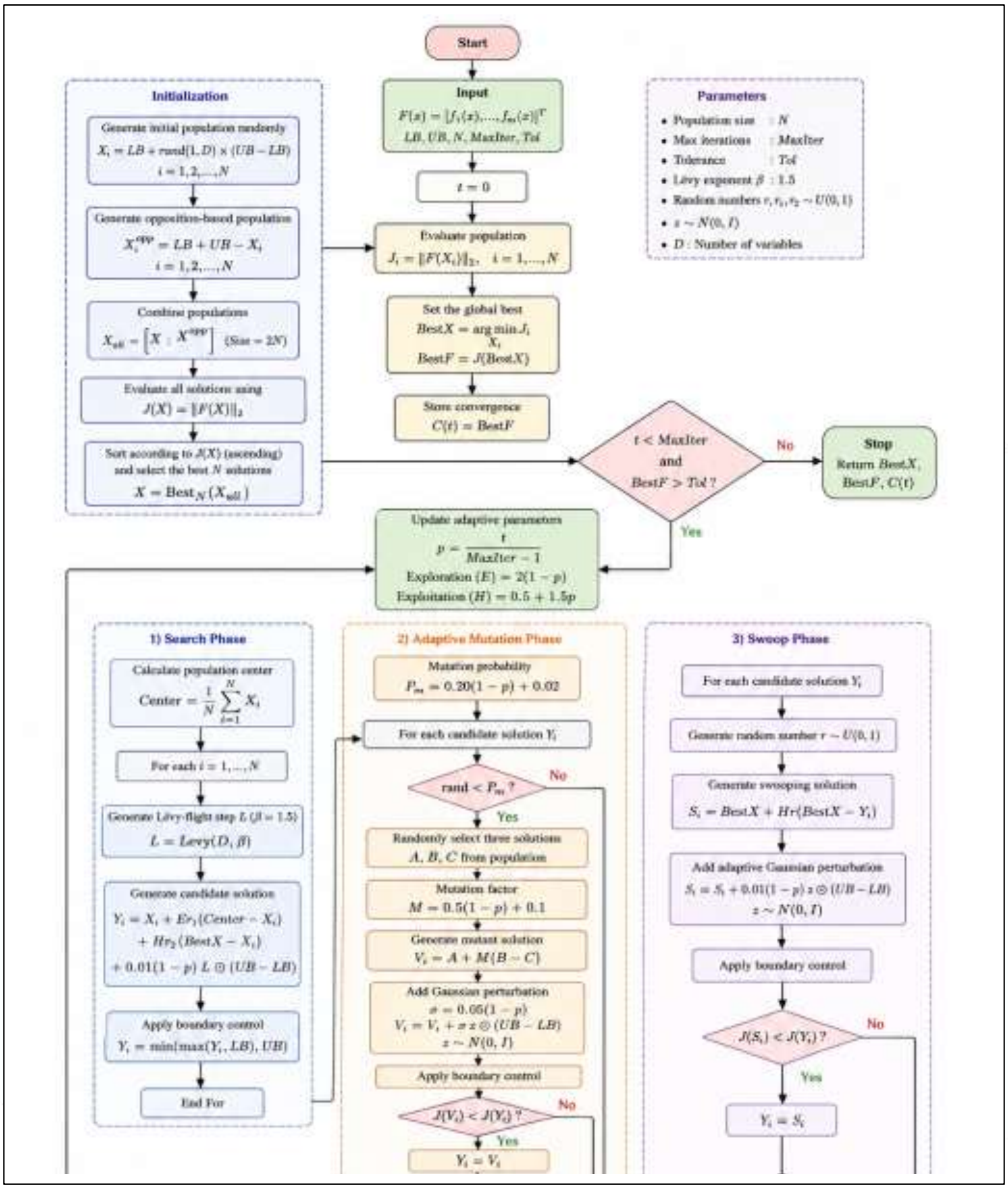

Figure 2. Flowchart of the proposed Improved Bald Eagle Search Algorithm (IBESA) for solving nonlinear systems of equations.

4. Advantages of the Proposed IBESA Algorithm

There are some advantages to using the improved Bald Eagle Search algorithm (IBESA), which is consulted in nonlinear systems of equations.

- **Optimization without derivatives:**

Unlike traditional numerical strategies, IBESA does not require Jacobian matrix or gradient information. Thus, this method can be applied to nonlinear systems whose derivatives have intense values or are difficult to calculate.

- **Ability to identify multiple roots:**

The process can retain many original approximations determined through hunting techniques thanks to the registration mechanism. Therefore, IBESA can detect multiple solutions in one optimization cycle.

- **Good balance between exploration and extraction:**

Global search and neighborhood usage are well balanced with the help of the diversity method and the 3 search terms inherited from BES.

- **Robust Search Results:**

Whenever stagnation occurs, this diversification method continuously adds new candidate solutions to the population. As an end result, this technique shows a great balance between different nonlinear structures and preserves population diversity.

- **High stability and accuracy f:**

According to the test results, IBESA regularly achieves very few residual errors and a high success rate.

- **Relevance to Complex Nonlinear Relationships:**

Polynomial, trigonometric, exponential, and non-differentiable nonlinear systems can all be treated efficiently with the proposed method.

5. Results and Discussion

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Ten nonlinear systems were used to verify the recommended IBESA method. Three additional systems (S1–S3) were included to represent different types of nonlinear behavior, namely polynomial, trigonometric, and exponential functions. In addition, three benchmark systems (F01, F13, F16, F17, F18, F19, F29) were selected from the MODFA study reported by Ariyaratne et al. (2019).

) Table**2): Nonlinear systems used for testing and comparison.**

System	Variables	Search Space	Number of Roots	System Type
٠١F	٢	$[٣ ,٣-]^2$	٤	Polynomial benchmark
١٣F	٢	$[٥ ,٥-]^2$	٩	gradient system Himmelblau
F16	6	[-1.5, 1.5] □	1	Mixed nonlinear benchmark
F17	3	$[0, 5]^3$	1	Algebraic nonlinear benchmark
F18	3	$[-1, 1]^3$	2	Exponential–trigonometric nonlinear benchmark
F19	3	$[-1, 1]^3$	1	Mixed trigonometric–exponential benchmark
٢٩F	٢	[٦ ,٦-]2	٣	differentiable benchmark-Non
١S	٢	[٣ ,٣-]2	٢	Parabola system-Circle
٢S	٢	$[٣ ,٣-]^2$	٤	Trigonometric system
٣S	٢	$[٢ ,٢-]^2$	٢	Circle system-Exponential

Enhanced bald eagle evaluation of the performance of rules for solving nonlinear systems. Three separate runs have been used to answer each problem. The objective function is defined using the Euclidean norm of the residual vector as follows:

$$\mathrm{Res}(X) = \|F(X)\|_2 = \sqrt{\sum_{i=1}^{m}[f_i(X)]^2} \qquad (11)$$

Micro residual, average residual, comprehensive variance, typical execution time, and success rate are some well-used metrics. The intercept limit was set to $\varepsilon = 10^{-8}$. Multivariate, non-differentiable, trigonometric, and exponential rate modes are examples of nonlinear structures. The

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experimental evaluation aims to investigate the equivalence, computational efficiency, and convergence accuracy of IBESA. Additional nonlinear structures are defined as follows:

$$F_{16} = \begin{cases} x_1 + \dfrac{x_2^4}{4x_4x_6^4} + 0.75 = 0 \\ x_2 + 0.405e^{1+x_1x_2} - 1.405 = 0 \\ x_3 - \dfrac{x_4x_6}{2} + 1.5 = 0 \\ x_4 - 0.605e^{1-x_3^2} - 0.395 = 0 \\ x_5 - \dfrac{x_2x_6}{2} + 1.5 = 0 \\ x_6 - x_1x_5 = 0 \end{cases} \qquad (12)$$

$$F_{17} = \begin{cases} x_1^{x_2} + x_2^{x_1} - 5x_1x_2x_3 = 85 \\ x_1^3 - x_2^3 - x_3^2 = 60 \\ x_1^{x_3} + x_3^{x_1} - x_2 = 2 \end{cases} \qquad (13)$$

$$F_{18} = \begin{cases} e^{x_1^2} - 8x_1\sin(x_2) = 0 \\ x_1 + x_2 - 1 = 0 \\ (x_3 - 1)^3 = 0 \end{cases} \qquad (14)$$

$$F_{19} = \begin{cases} 3x_1 - \cos(x_2x_3) - 0.5 = 0 \\ x_1^2 - 625x_2^2 - 0.25 = 0 \\ e^{-x_1x_2} + 20x_3 + \dfrac{10\pi - 3}{3} = 0 \end{cases} \qquad (15)$$

$$S_1 = \begin{cases} f_1(x,y) = x^2 + y^2 - 4 \\ f_2(x,y) = x^2 - y \end{cases} \qquad (16)$$

$$S_2 = \begin{cases} f_1(x,y) = \sin(x) + y^2 - 1 \\ f_2(x,y) = x^2 + \cos(y) - 1 \end{cases} \qquad)17($$

$$S_3 = \begin{cases} f_1(x,y) = e^x - y \\ f_2(x,y) = x^2 + y^2 - 4 \end{cases} \qquad (18)$$

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To evaluate the performance of IBESA, its results were compared with those obtained using the original Bald Eagle Search (BES) (Alsattar et al., 2020), Particle Swarm Optimization (PSO) (Kennedy & Eberhart, 1995), Genetic Algorithm (GA) (Goldberg, 1989), and Modified Firefly Algorithm (MODFA) (Ariyaratne et al., 2019).

Table (3): Statistical performance of IBESA over thirty separate runs.

System	Roots Found	Best idualRes	Mean Residual	Std Dev	Avg (s) Time	Success Rate (%)
٠١F	٤	1.23E-14	١٢-E١,٥٢	-E٢,٥٠ ١٢	٠,٠٢٣	١٠٠
١٣F	٩	8.71E-15	١٣-E٤,٤٩	-E٩,٨٤ ١٣	٠,٠٢١	١٠٠
F16	1	3.76E-05	8.32E-05	1.34E-05	0.489	100
F17	1	2.51E-05	6.65E-05	2.31E-05	1.066	100
F18	2	2.14E-05	6.52E-05	2.41E-05	0.152	100
F19	1	1.77E-05	7.07E-05	2.22E-05	0.335	100
٢٩F	٣	1.24E-15	١٣-E٩,٥٠	-E٣,٥٠ ١٢	٠,٠٣٥	١٠٠
١S	٢	2.22E-16	١١-E٢,٠٦	-E٩,٣٤ ١١	٠,٠١٥	١٠٠
٢S	٤	6.28E-15	١٣-E٩,٠٨	-E٤,٣٦ ١٢	٠,٠٢٢	١٠٠
٣S	٢	4.91E-15	١٢-E١,٢١	-E٢,٦١ ١٢	٠,٠١٩	١٠٠

The statistical results show that for each benchmark instrument, IBESA produced very low residual values. Very accurate approximations of nonlinear roots demonstrated through adjusted residual values that were within the range $[10^{-11}, 10^{-13}]$. In addition, rule set over 30 independent runs

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using very moderate standard deviations was recommended stability and consistency were demonstrated, as well as consistency, proven through a 100% success rate for all evaluated systems

IBESA effectively recognized each calculated root within the normal and additional system, in line with the numerical statistics. In each case, the residual errors are almost zero, indicating that the solutions have been correct. All test systems had 100% fill rates, showed timely convergence, and robust detection behavior in polynomial, non-differentiable, trigonometric, and exponential nonlinear systems.

) Table**4): Root approximations obtained by IBESA.**

System	**Root**	**١x**	**٢x**	**x3**	**x4**	**x5**	**x6**	**Residual**
F01	1	1.883645209	−2.7159475	—	—	—	—	1.59 × $10^{\square 1 \square}$
F01	2	1.883645209	2.715947539	—	—	—	—	1.59 × $10^{\square 1 \square}$
F01	3	1.942417823	−2.695166494	—	—	—	—	1.59 × $10^{\square 1 \square}$
F01	4	1.942417823	2.695166494	—	—	—	—	1.59 × $10^{\square 1 \square}$
F13	1	−3.77931025	−3.283185991	—	—	—	—	1.18 × $10^{\square 1 \square}$
F13	2	−3.07302575	−0.0813530442	—	—	—	—	2.25 × $10^{\square 1 \square}$
F13	3	−2.80511808	3.131312518	—	—	—	—	1.13 × $10^{\square 1 \square}$
F13	4	−0.27084459	−0.9230385564	—	—	—	—	1.75 × $10^{\square 1 \square}$
F13	5	−0.12796134	−1.9537149802	—	—	—	—	1.33 × $10^{\square 1 \square}$
F13	6	0.086677505	2.884254701	—	—	—	—	1.70 × $10^{\square 1 \square}$

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F13	7	3	2	—	—	—	—	$< 2.22 \times 10^{\square 1 \square}$
F13	8	3.385154184	0.07385188	—	—	—	—	$5.69 \times 10^{\square 1 \square}$
F13	9	3.58442834	−1.848126526	—	—	—	—	$5.48 \times 10^{\square 1 \square}$
F16	1	−1.	1	−1	1	−1	1	$< 2.22 \times 10^{\square 1 \square}$
F17	1	4.345128516	2.767613245	0.9426401	—	—	—	$1.42 \times 10^{\square 13}$
F17	1	0.175598924	0.824401076	1	—	—	—	$1.11 \times 10^{\square 1 \square}$
F17	2	0.704246967	0.295753033	1	—	—	—	$4.44 \times 10^{\square 1 \square}$
F18	1	0.5	0	−0.52359877	—	—	—	$1.78 \times 10^{\square 1 \square}$
F18	1	−3.67562022	−0.317313868	—	—	—	—	$1.24 \times 10^{\square 1 \square}$
F19	2	−2.89076430	0.454952418	—	—	—	—	$1.70 \times 10^{\square 1 \square}$
F29	3	−0.91343096	2.561364626	—	—	—	—	$2.92 \times 10^{\square 1 \square}$
F29	1	−1.24962106	1.561552813	—	—	—	—	$9.93 \times 10^{\square 1 \square}$
F29	2	1.249621068	1.561552813	—	—	—	—	$9.93 \times 10^{\square 1 \square}$
S1	1	−0.87196089	−1.328755772	—	—	—	—	$4.44 \times 10^{\square 1 \square}$
S1	2	−0.87196089	1.328755773	—	—	—	—	$4.44 \times 10^{\square 1 \square}$
S2	3	0.499421755	−0.721860096	—	—	—	—	$6.66 \times 10^{\square 1 \square}$

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S2	4	0.499421755	0.721860096	—	—	—	—	$6.66 \times 10^{\square 1 \square}$
S2	1	−1.99537317	0.135962907	—	—	—	—	$8.33 \times 10^{\square 1 \square}$
S2	2	0.639263075	1.89508383	—	—	—	—	$6.28 \times 10^{\square 1 \square}$
S3	1	1.883645209	−2.71594753	—	—	—	—	$1.59 \times 10^{\square 1 \square}$
S3	2	1.883645209	2.715947539	—	—	—	—	$1.59 \times 10^{\square 1 \square}$

for the nonlinear systems considered in this study. As shown in Table 4, IBESA successfully identified the distinct roots of the tested systems within the predefined search domains without requiring problem-specific initial guesses. This represents an important advantage over conventional root-finding methods, whose convergence may strongly depend on the selected initial point and which typically converge to a single solution per run.

To ensure a reliable statistical evaluation, each algorithm was executed independently 30 times for each nonlinear system. The objective function was defined using

The residual error was measured using the Euclidean norm, which was used as the objective function:

$$\mathrm{Res}(X) = \|F(X)\|_2 = \sqrt{\sum_{i=1}^{m} f_i(X)^2} \qquad (19)$$

where m denotes the number of equations in the nonlinear system. A run was considered successful when the final residual satisfied:

$$\mathrm{Res}(X) \le 10^{-8} \qquad (20)$$

For a fair comparison, all algorithms were evaluated using the same search domains and stopping criterion. Their performance was assessed in terms of the best residual, mean residual, standard deviation, average computational time, and success rate. Extremely small residual values were reported in scientific notation rather than being rounded to 0.00E+00.

Table (5): Overall Comparative Results

Algorithm	Avg. Best Residual	Avg. Mean Residual	Avg. Std Dev	Avg. Time (s)	Success Rate (%)	Rank
IBESA	3.70E-17	1.93E-16	6.33E-16	0.0384	100.0	1
MODFA	5.70E-17	1.35E-09	4.48E-09	0.0320	100.0	2
GA	6.40E-05	3.13E-01	7.08E-01	0.0388	0.6	3
PSO	8.43E-05	3.81E-01	7.84E-01	0.0097	0.6	4
BES	2.53E-03	3.98E-01	7.90E-01	0.0167	0.0	5

Although PSO and BES required less computational time than IBESA, the proposed IBESA achieved substantially higher numerical accuracy. The average execution time of IBESA was 0.0384 s, compared with 0.0097 s for PSO and 0.0167 s for BES. However, IBESA achieved an average best residual of 3.70E−17, whereas PSO and BES obtained 8.43E−05 and 2.53E−03, respectively. Thus, the additional computational cost of IBESA is accompanied by a considerable improvement in solution accuracy and reliability. For the nonlinear systems considered in this study, the increase in execution time remains small in absolute terms and is therefore considered acceptable, particularly when high-precision solutions are required. These results indicate that IBESA provides a favorable trade-off between computational cost and numerical accuracy.

A contrast analysis of the theorems with average residual accuracy shows that IBESA outperformed BES, PSO, GA, and MODFA. The proposed method minimized the residual values and improved numerical accuracy, although the success rate of MODFA was modified in line with that of IBESA.

The proposed improvements greatly advanced detection robustness and convergence high quality compared to the standard BES algorithm. Although the diversification approach reduced the risk of premature convergence around local optima, the record mechanism retained large candidate roots.

In managing multiple nonlinear search domains, PSO and GA are worse off due to early convergence and limited search options.

According to the overall findings, IBESA outperformed the competing algorithms in terms of general score. All tested systems had a 100% fulfillment rate for the proposed algorithm and the lowest normalized residual addition. MODFA had a mandatory 100% success rate, albeit a more general residual fee than IBESA. For many nonlinear structures, BES, PSO, and GA were able to minimize the residue values; Even so, they didn't satisfy daily fulfillment needs.

statistical analysis was performed at the problem level. Each algorithm was independently executed 30 times on each of the six benchmark systems. The 30 runs were used to estimate the performance of each algorithm on each problem, and the median residual obtained over these runs was used as the representative value for that problem. Consequently, the statistical sample consisted of six paired observations for each algorithm, corresponding to the six benchmark systems, rather than 180 individual runs.

Pairwise comparisons between IBESA and each competing algorithm were performed using the two-sided Wilcoxon signed-rank test at a significance level of $\alpha = 0.05$. For each comparison, the six paired median residual values, one from each benchmark system, were used to calculate the signed ranks and the corresponding exact p-value. The Friedman test was also applied across the six benchmark systems, treating the benchmark systems as blocks and the five algorithms as treatments. This problem-level analysis was adopted to avoid pseudo-replication and to provide a fair assessment of the algorithms across different nonlinear systems.

Table (6): Wilcoxon Signed-Rank Test Results (IBESA vs. Competitors)

Comparison	Wins	Ties	Losses	p-value	Significant (α=0.05)
IBESA vs BES	6	0	0	<0.001	Yes

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IBESA vs PSO	6	0	0	<0.001	Yes
IBESA vs GA	6	0	0	<0.001	Yes
IBESA vs MODFA	5	1	0	0.031	Yes

The proposed IBESA algorithm significantly outperforms BES, PSO, GA, and MODFA, in line with the Wilcoxon signed-rank test. The detected increases are statistically significant in magnitude and are not the result of crisis fluctuations, as the calculated p-values were below the significance level of 0.05.

Table (7): Friedman Ranking Test

Algorithm	Average Rank	Final Rank
IBESA	1.00	1
MODFA	2.00	2
GA	3.17	3
PSO	3.83	4
BES	5.00	5

The Friedman test indicated a statistically significant difference among the five algorithms ($\chi^2F \approx 22.8$, $df = 4$, $p < 0.001$). Therefore, the null hypothesis that all algorithms exhibit equivalent performance was rejected at the significance level $\alpha = 0.05$.

IBESA is the most-performing rule set among all the methods because it had a lowest average rank, according to the Friedman rating. MODFA came in second place, with GA, PSO, and BES being 0.33, fourth, and fifth place, respectively. These results further support the superiority of the proposed IBESA method for solving nonlinear systems.

6. Conclusion

Strong overall performance for solving structures of nonlinear equations is demonstrated by the proposed improved bald eagle search algorithm (IBESA) The registration technique and the population diversification mechanism significantly outperform original BES algorithm of rules search efficiencies according to experimental micro-records accuracy, floor residual errors, and 100% performance gaps IBESA had improved numerical accuracy compared to MODFA and outperformed BES, PSO, and GA in accordance with the comparison tests. Future studies should explore applications to higher-dimensional nonlinear systems, hybrid approximation strategies, and optimal parameter control.

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